Dynamic optimal control for distress large financial networks and Mean field systems with jumps

Rui Chen

THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres
PSL Research University

Préparée à Université Paris-Dauphine

Dynamic optimal control for distress large financial networks and Mean field systems with jumps

École doctorale n°543

ÉCOLE DOCTORALE DE DAUPHINE

Spécialité MATHÉMATIQUE APPLIQUÉES

COMPOSITION DU JURY :

Huyên Pham
Université Paris Diderot
Président

Agnès Sulem
INRIA de Paris
Directeur de thèse

Aurélien Alfonsi
École des Ponts ParisTech
Rapporteur

Luitgaard Veraart
London School of Economics
Rapporteur

Guillaume Carlier
Université Paris Dauphine
Examinateur

Andreea Minca
Cornell University
Examinateur

Soutenue par **Rui CHEN**
le 19-07-2019

Dirigée par **Agnès Sulem**

DAUPHINE UNIVERSITÉ PARIS | PSL RESEARCH UNIVERSITY PARIS

UNIVERSITÉ PARIS-DAUPHINE

École doctorale **École Doctorale Sciences Mathématiques de Paris Centre**

Unité de recherche **Centre De Recherche en Mathématiques de la Décision, Université Paris-Dauphine**

Thèse présentée par **Chen RUI**

Soutenue le **19 juillet 2019**

En vue de l'obtention du grade de docteur de l'Université Paris-Dauphine

Discipline **Mathématique Appliquées**

Spécialité **Mathématique Financière**

Dynamic optimal control for distress large financial networks and Mean field systems with jumps

Thèse dirigée par Agnès SULEM

Composition du jury

Rapporteurs	Aurélien ALFONSI	professeur au École des Ponts ParisTech
	Luitgaard VERAART	professeur au London School of Economics
Examinateurs	Huyên PHAM	professeur à l'Université Paris Diderot
	Guillaume CARLIER	professeur à l'Université Paris Dauphine
	Andreea MINCA	professeur au Cornell University
Directeur de thèse	Agnès SULEM	directeur de recherche à l'INRIA

Mots clés : risque systémique, réseaux financiers, graphes aléatoires, contagion de défauts, contrôle optimal, edsrs à champs moyen avec sauts.

Keywords: systemic risk, financial networks, random graphs, default contagion, optimal control, bsdes with jumps, mean field bsdes.

Cette thèse a été préparée au

Centre De Recherche en Mathématiques de la Décision, Université Paris-Dauphine

Place du Maréchal De Lattre De Tassigny
75016 Paris
France

☎ +44 7826274502
Site http://ceremade.dauphine.fr

CEREMADE
UMR CNRS 7534

To my family

Résumé

Cette thèse propose des modèles et des méthodes pour étudier le contrôle du risque dans de larges systèmes financiers. Nous proposons dans une première partie une approche structurelle : nous considérons un système financier représenté comme un réseau d'institutions connectées entre elles par des interactions stratégiques sources de financement mais également par des interactions qui les exposent à un risque de contagion de défaut. La nouveauté de notre approche réside dans le fait que ces deux types d'interaction interfèrent. Nous proposons des nouvelles notions d'équilibre pour ces systèmes et étudions la connectivité optimale du réseau et le risque systémique associé.

Dans une deuxième partie, nous introduisons des mesures de risque systémique définies par des équations différentielles stochastiques rétrogrades dirigées par des opérateurs à champ moyen et étudions des problèmes d'arrêt optimal associés. La dernière partie aborde des questions de liquidation optimale de portefeuilles.

Mots clés : Risque systémique, réseaux financiers, graphes aléatoires, contagion de défauts, contrôle optimal, EDSRs à champs moyen avec sauts.

Abstract

This thesis presents models and methodologies to understand the control of systemic risk in large systems. We propose two approaches. The first one is structural: a financial system is represented as a network of institutions. They have strategic interactions as well as direct interactions through linkages in a contagion process. The novelty of our approach is that these two types of interactions are intertwined themselves and we propose new notions of equilibria for such games and analyze the systemic risk emerging in equilibrium. The second approach is a reduced form. We model the dynamics of regulatory capital using a mean field operator: required capital depends on the standalone risk but also on the evolution of the capital of all other banks in the system. In this model, required capital is a dynamic risk measure and is represented as the solution of a mean-field BDSE with jumps. We show a novel dual representation theorem. In the context of mean-field BSDEs the representation gives yield to a stochastic discount factor and a worst-case probability measure that encompasses the overall interactions in the system. We also solve the optimal stopping problem of dynamic risk measure by connecting it to the solution of reflected mean-field BSDE with jumps. Finally, We provide a comprehensive model for the order book dynamics and optimal Market making strategy appeared in liquidity risk problems.

Keywords: Systemic risk, financial networks, random graphs, default contagion, optimal control, BSDEs with jumps, Mean field BSDEs.

Contents

1 Overview **1**
1 Summary 3
2 Optimal connectivity for a large dynamic financial network subject to contagion risk 4
3 Optimization problems for Mean-Field BSDEs with jumps 5
4 Optimal inventory management and order book modeling 8
5 Publications and Working Papers 12

2 Optimal connectivity for a large dynamic financial network subject to contagion risk **13**

I Network connectivity as equilibrium in a large game **15**
1 Node performance under linkage benefits and contagion risk 15
2 Nodes' optimal connectivity choice 23
2.1 The static case ; No recovery feature ($\alpha = 0$) 23
2.2 The dynamic threshold case ($\alpha > 0$) 25
3 Equilibrium 26
3.1 Analysis of the equilibrium 29
4 Numerical results 31
5 Conclusions 32
6 Proofs and asymptotic results 32
6.1 A Markov Chain Description of Contagion Dynamics 33
6.2 A Law of Large Numbers for the Contagion Process 36
6.3 Proof of Theorem 1.2 39

II Extensions : Different recovery intensities and policies **47**
1 Different recovery intensities modeling between two interactions 47
2 Different growth attribution policies 48
2.1 Growth Policy I 48
2.2 Growth Policy II 52

Bibliography **55**

3 Optimization problems for Mean-Field BSDEs with jumps **57**

III Introduction **59**

IV Mean-field BSDE and application to Global Dynamic Risk Measures **63**
1 Mean-field BSDEs with jumps 63
1.1 Notation and definitions 63
1.2 Comparison Results 65
2 Global dynamic risk measures 69

2.1 Definition and properties . . . 69
2.2 Dual representation of convex global risk measures . . . 72
3 Optimization principle for Mean-field BSDEs . . . 79

V Optimal stopping for Mean-Field BSDEs with jumps 81
1 Reflected Mean-Field BSDEs with jumps . . . 81
1.1 Notation and definitions . . . 81
1.2 Comparison theorems for Mean-Field RBSDEs with jumps . . . 82
2 Optimal stopping for global dynamic risk measures . . . 84
3 Optimization principles for Reflected Mean-Field BSDEs . . . 88
4 Appendix . . . 92

VI Optimal stopping for Global risk measures in the case of multiple priors 97
1 Robust optimal stopping problem . . . 97
2 Application to the case of multiple priors . . . 101

VII Another type of Mean-field BSDEs and related results 105
1 Mean-field BSDEs with jumps . . . 105
2 Optimal stopping for Mean-Field BSDEs with jumps . . . 107
3 Robust optimal stopping problem . . . 112
4 Appendix . . . 114

Bibliography 121

4 Optimal inventory management and order book modeling 123

VIII Market maker strategy 127
1 The order book dynamics . . . 127
2 The Market Maker dynamics and Control set . . . 128
3 Transition states of the reduced state process X^{ζ} . . . 129
4 The optimal control problem . . . 133
5 Numerical resolution . . . 136

IX Optimal high frequency strategy and VWAP strategy 139
1 An optimal high frequency strategy: Pair trading and impulse control . . . 139
1.1 Impulse control on the dynamics of the spread . . . 139
1.2 Dynamic programming principle and quasi-variational Hamilton-Jacobi-Belman equation . . . 140
1.3 Derivation of the optimal control and Numerical solution . . . 141
2 A VWAP based strategy for portfolio liquidation . . . 143
2.1 Discrete time framework - VWAP based strategy . . . 144
2.2 Time continuous framework . . . 146

X Market simulation 149

XI Appendix **151**

1 Ergodicity of the Order book dynamics 151
2 Numerical scheme error estimate 153
3 Model parameters estimation 156
3.1 Intensities estimation 156
3.2 Regeneration process 157

Bibliography **159**

Part 1

Overview

1 Summary

In Part one, we investigate the optimal choice of connectivity and define a new notion of equilibrium for a set of financial institutions in a large financial network. Players are represented as nodes, linkages are source of income, and at the same time they bear the risk of contagion. The optimal connectivity of the nodes results from a game, in which the risk of contagion depends on the choices of all nodes in the system. The players' payoff is impacted by their own choice of connectivity, which governs both the benefits from linkages as well as the exposure to the risk of contagion in the network. The risk of contagion in turn depends on all the players' connectivity choices. Our notion of equilibrium is similar to a mean field game: players take as given a mean-field, namely the conjectured failure probability of a link (which also gives the proportion of failed nodes at the end of a potential contagion process). They then decide on their own connectivity. This leads to an actual failure probability in the network and we check that a fixed point holds: the actual link failure probability is the same as the conjectured link failure probability.

We analyze the existence of equilibrium when contagion is driven by the classical threshold contagion model. Moreover, we provide several generalizations of this model to the dynamic case. First, we allow nodes to receive growth benefits uniformly over time during the contagion processes. Second, we investigate policies by which growth is distributed more efficiently over nodes and time depending on the state. We call these recovery features and we furthermore investigate the optimal connectivities in equilibrium under such recovery features. Our results show that a higher heterogeneity in the initial distribution of the threshold (as captured by its standard deviation) implies a lower default probability in equilibrium. A higher heterogeneity in the initial distribution of the threshold also leads to a larger average connectivity in equilibrium. Finally, systems with higher growth/recovery rates may lead to equilibria with higher failure probability as well as higher final fraction of failed banks. This result is surprising and gives new insights into potential policies that promote financial stability. In particular, this shows that even if equity is injected over time, the strategic banks will adapt and potentially take more risks in equilibrium. This means that any policies that promote growth must be accompanied by limiting connectivity and this must be targeted on banks which have higher initial thresholds. Otherwise, such banks, in anticipation of future growth would otherwise take too high risks. By limiting their connectivity, their thresholds will grow even larger and they will act as shock absorbs of the system.

In Part two we study Mean-Field BSDEs with jumps and the corresponding Reflected Mean-field BSDE. We first introduce and show the existence and uniqueness of two types of mean-field BSDEs with jumps. We provide (strict) comparison results and a dual representation results with application to convex dynamic risk measure. We then provide the same stream of results for the Reflected Mean-field BSDEs with jumps. We solve the optimal stopping problem of risk measure by characterizing solution of optimal stopping problem of mean field- BSDE as the solution of Reflected mean-field BSDEs with jumps. Finally we provide results on the optimization principle and robust optimal stopping problems with applications

In Part three we study the problem of the order book dynamics modeling and market makers optimal strategy. We present the a comprehensive Queue-Reactive type model got order book dynamics. We then derive the HJB variational partial differential equations satisfied by the value functions of the market maker who is trying to optimize the expected utility of terminal wealth. We propose a numerical solution for this problem and illustrate how approximated optimal strategy can be deduced from them.

2 Optimal connectivity for a large dynamic financial network subject to contagion risk

Interconnected systems are subject to contagion in time of distress. Recent effort has been dedicated to understanding the relation between network topology and the scope of distress propagation, see e.g., [1, 2, 5, 17] and the references therein. It is critical to recognize that connectivity is a result of an optimization problem of agents, who derive benefits from connections and view the associated contagion risk as a cost. In the context of financial systems, this benefit-cost view of connections is first presented in [5], who analyze socially optimal network topologies.

A threshold model of contagion for heterogenous graphs is proposed in [1]. Each node represents a bank, insurance company or institution, and is endowed with a threshold to contagion. A set of nodes fail exogenously and any node in the system can fail due to contagion when its threshold is surpassed by the number of connected nodes that fail. The availability of asymptotic results on the fraction of nodes that fail at the end of the contagion process allows us to estimate the failure probability for a network with given connectivity. In this thesis, we take a step further and find the equilibrium choice of connectivity. The banks' payoff is impacted by their own choice of connectivity, which governs both the benefits from linkages as well as the exposure to the risk of contagion in the network. The risk of contagion in turn depends on all the banks' connectivity choices.

The fundamentals in the model are represented by the thresholds. Under full information, the distribution of the thresholds across banks is common knowledge. Once a node chooses its connectivity, the network is generated using the configuration model. This is a random graph chosen uniformly over all graphs with the given connectivity sequence across nodes. For this random graph, a law of large numbers gives the limit probability that a randomly chosen link fails, i.e. it links to a failed node. This probability depends on the join distribution of the connectivity and thresholds. Our equilibrium concept is a rational expectations equilibrium. For an anticipated link failure probability, nodes choose their optimal connectivity as a function of their threshold. We then ensure that the anticipated link failure probability is equal to the failure probability in the network with the chosen connectivity.

Our equilibrium model is quite different from past literature. In [5] the authors use the notion of pairwise Nash stability and a contagion mechanism in which nodes fail with an exogenous probability when a neighbor fails. The notion of pairwise Nash stability is also used in [9]. Here we exploit the asymptotic results for contagion scope in our particular choice of random graph, where connectivity is optimized but the actual neighbors are randomly chosen. We are interested in the optimal connectivity as a function of fundamentals, namely the thresholds. Our notion of equilibrium can be seen as a Nash equilibrium with a continuum of players, classified according to their thresholds and we obtain a unique such equilibrium.

For the asymptotic case, our results allow us to understand how the degree of heterogeneity in the fundamentals translates into failure risk in the network with endogenous connectivity. Nodes adjust their connectivity to the degree of heterogeneity in the fundamentals. When the fundamentals are more homogenous, then the average connectivity is lower, but the failure probability is higher than in the case when there is more dispersion in the fundamentals. The network payoff is defined as the average number of links times the survival probability. For different fundamentals, the network connectivity is adjusted in equilibrium to yield the same network payoffs. However, these networks are not equivalent from the point of view of systemic risk. To obtain lower default probability and systemic risk, more dispersion in fundamentals is preferable. In this case, nodes with large thresholds would act as stabilizers in time of distress. Our results are in line with [9], who obtain in a different model that economies that are fundamentally "safer", in the sense that they are subject to less volatile shocks, generate higher interconnectedness. Here, it is the homogeneity of the thresholds, or otherwise said their lesser variance, that generates more connectivity.

Our work is complementary to the line of research on the control of contagion, e.g., [3, 15, 16]. In these works, a central party, for example a regulator or government, seeks to minimize contagion. In [8] the author explores the effect of moral hazard on network topology, as the network is formed with the anticipation of government bailouts in case of distress. Parallel to the development of network models for systemic risk, a recent series of works [1, 4, 4–6] introduced a reduced form approach to systemic risk analysis, based on mean field interaction models. In these works, the trajectories of the banks are modeled as a set of coupled diffusions, which may be controlled by a central party that specifies the parameters of the interaction. Here in contrast, the financial institutions themselves are the decision makers, and their decision is made before the shock, with a rational expectation on the way the cascade will evolve following the shock.

While most of the work on contagion in financial networks refers to banking networks, a recent literature [4, 14] considers insurance-reinsurance networks. Contagion proceeds similarly in such networks, as failed reinsurers cannot honor contracts to other institutions, and such failures can propagate via chains of reinsurance contracts. Our work offers thus guidance on the formation of such networks and their inherent risk.

3 Optimization problems for Mean-Field BSDEs with jumps

Mean-Field BSDE is firstly introduced and studied by Buckdahn *et al* [9] [10] and by Li *et al* [8] for the case with jumps.

Mean-Field BSDEs The Mean-field BSDE is a process $(X, Z, l) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ satisfying

$$\begin{cases} -dX_t = \mathbb{E}'\left[f(t,\omega,X'_t,Z'_t,l'_t(\cdot),X_t,Z_t,l_t(\cdot))\right]dt - Z_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt,de); \\ X_T = \xi. \end{cases} \tag{1}$$

where $\mathbb{E}' f(t,\omega,X'_t,Z'_t,l'_t(\cdot),X_t,Z_t,l_t(\cdot)) = \int_\Omega f(t,\omega,X_t(\omega'),Z_t(\omega'),l_t(\cdot)(\omega'),X_t(\omega),Z_t(\omega),l_t(\cdot)(\omega))d\mathbb{P}(\omega')$

The solution of the Mean-field BSDE (1) was related to the solution (X^N, Z^N, l^N) of the backward equation

$$\begin{cases} -dX_t = \frac{1}{N}\sum_{j=1}^{N}\left[f(t,\omega,X_t^{j,N},Z_t^{j,N},l_t^{j,N}(\cdot),X_t,Z_t,l_t(\cdot))\right]dt - Z_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt,de); \\ X_T = \xi. \end{cases} \tag{2}$$

where the i.i.d. sequence $(X^{j,N},Z^{j,N},l^{j,N}),\ 1 \leq j \leq N$, are following the same law as (X,Z,l). This convergence result is proved in [9] in the case of Brownian motion, namely they show (X,Z,l) to be the uniform limit of the solution (X^N,Z^N,l^N) when $N \to \infty$.

It is typical in the literature to motivate the study of the Mean-field BSDE (1) by considering it as an approximation of a system of N coupled symmetric stochastic differential equations

$$\begin{cases} -dX_t^{i,N} = \frac{1}{N}\sum_{j=1}^{N} f(t,\omega,X_t^{j,N},X_t^{i,N})dt - Z_t^{i,N}dW_t - \int_{\mathbf{E}} l_t^{i,N}(e)\tilde{N}(dt,de); \\ X_T^{i,N} = \xi. \end{cases}$$

On the other hand, Reflected backward stochastic differential equations (RBSDEs in short) have been introduced in 1997 by the five authors El Karoui, Kapoudjian, Pardoux, Peng and Quenez [3] in the case of a filtration generated by the Brownian motion. The generalization to the case of reflected BSDEs with jumps, which is a standard reflected BSDE driven by a Brownian motion and an independent Poisson random measure, has been established by Hamadene and Ouknine in [15]. A solution for such equation, associated with a coefficient f, terminal value η and a barrier ξ, is a quadruple of process $(Y,Z,k(.),A)$ of adapted solutions which satisfy the following equation:

$$\begin{cases} -dY_t = f(t,\omega,X_t,Z_t,k_t(\cdot))dt + dA_t - Z_t dW_t - \int_U k_t(u)\tilde{N}(dt,du); \quad Y_T = \eta, \\ Y_t \geq \xi_t,\ 0 \leq t \leq T \text{ a.s.}, \\ \int_0^T (Y_t - \xi_t)dA_t = 0 \text{ a.s. }; \end{cases} \tag{3}$$

where A is a continuous, increasing process whose role is to push the solution Y such it remains above the barrier ξ. The condition $\int_0^T (Y_t - \xi_t)dA_t = 0$ ensures that the process A acts in a minimal way. More precisely, A increases only on the set $\{Y = \xi\}$. The authors have shown the existence and uniqueness of solutions under the assumption barrier ξ is right continuous left- limited (RCLL) whose jumping times are inaccessible stopping times. In this case, the jumping times of the process Y come only from those of its Poisson process and then they are inaccessible. The general case of RBSDEs with jumps and irregular obstacles has been studied more recently by Quenez-Sulem [12] which weaken the assumption on ξ to be only RCLL. This allows the jumping times of process Y come not only from those of its Poisson process (inaccessible jumps) but also from those of the process ξ (predictable jumps). The allowance of ξ to have predictable jumps also make the reflecting process A no longer continuous. the difference with respect to (3) only appears in the Skorokhod condition which becomes: $\int_0^T (Y_t - \xi_t)dA_t^c = 0$ a.s. and $\Delta A_t^d = -\Delta Y_t \mathbf{1}_{\{Y_{t-}=\xi_{t-}\}}$.

An important application of reflected BSDEs is its connection to optimal stopping problems of dynamics risk measure. More precisely, given an RCLL process $(\xi_t, 0 \leq t \leq T)$ and a Lipschitz driver f satisfying the additional assumption such that the comparison theorem holds, the solution Y of the associated RBSDE satisfies: for each stopping time $S \in \mathbb{T}_0$, we have

$$Y_S = \text{ess}\sup_{\tau \in \mathbb{T}_S} X_S(\xi_\tau, \tau) \quad \text{a.s.} \tag{4}$$

where for $\tau \in \mathbb{T}_S$, $X.(\xi_\tau, \tau)$ is the solution of the BSDE associated with terminal time τ, terminal condition ξ_τ, and driver f. Meanwhile, optimization principle and robust optimal stopping problems has also been studied in [12] where the minimizer of a set of RBSDE solution $\{Y^\alpha, \alpha \in \mathcal{A}\}$ driven by $\{f^\alpha, \alpha \in \mathcal{A}\}$ is characterized as the solution of an RBSDE. Then robust optimal stopping problem turned to be a mixed control/optimal stopping game problem and the authors showed the existence of a value function for the game problem under some additional hypothesis.

Contributions In this chapter, we provide some results and properties for the Mean-field BSDE with general Mean-field operators and the corresponding the reflected Mean-field BSDE. We also generalized the results between BSDEs with RBSDEs to the case with Mean-field drivers. Then we apply the results to solve the problems of Mean-field dynamic risk measures including the (robust) optimal stopping problems.

We firstly consider the Mean-field BSDEs with general operator F :
A solution of a Mean-field BSDE with jumps with terminal time T, terminal condition ξ and driver f and operator F consists of a triple of processes $(X, Z, l) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ satisfying

$$\begin{cases} -dX_t = f(t, \omega, F(t, X_t(\cdot)), X_t, Z_t, l_t(\cdot))dt - Z_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt, de); \\ X_T = \xi. \end{cases} \tag{5}$$

where F is $\mathcal{B}([0,T]) \times \mathcal{B}(L^2_{\mathbf{P}})$ measurable operator from $[0,T] \times L^2_{\mathbf{P}}(\mathcal{F}_T)$ to $\mathbb{R}$.
We denote by $(X(\xi,T), Z(\xi,T), l(\xi,T))$ the solution of the Mean-field BSDE associated with terminal time T and (ξ, f). Under additional Lipschitz condition, we showed the existence/uniqueness and (strict) comparison results. We notice that in [14], the authors also considered similar Mean-field BSDEs, however, one of main difference we want to address is that we have showed results under more general Mean-field operator which includes cases with higher order interactions while in [14] the authors mainly consider the linear specially case $F(t, X) = \mathbb{E}[\varphi(t, X)]$.

We then introduce the functional $\rho : (\eta, T') \mapsto \rho.(\eta, T')$ representing a *mean-field dynamic risk measure* induced by the mean-field BSDE with driver f and mean-field operator F. To be more precise : Let $T > 0$ be a time horizon and f be a Lipschitz driver.

For each $T' \in [0,T]$ and $\eta \in L^2(\mathcal{F}_{T'})$, set

$$\rho_t^{f,F}(\eta, T') = \rho_t(\eta, T') := -X_t(\eta, T'), \quad 0 \le t \le T', \tag{6}$$

where $X_t(\eta, T')$ denotes the solution of mean-field the BSDE (5) with driver f, mean-field operator F and terminal conditions (T', η). If T' represents a given maturity and η a financial position at time T', then $\rho_t(\eta, T')$ is interpreted as the risk of η at time t. We provide properties of these Mean-field dynamic risk measures such as monotonicity, translation invariance, convexity under appropriate hypotheses using the properties we have proved for the solutions to Mean-field BSDEs. Meanwhile, we also give a technical dual representation results

$$\mathbb{E}\rho_t(\xi, T) = \sup_{(\gamma,\beta,q,\alpha)\in\bar{\mathcal{A}}_T} \left[\mathbb{E}^{\mathcal{Q}^\alpha} D^{\beta,\gamma}_{t,T}(-\xi) - \zeta(\gamma, \beta, q, \alpha, T)\right] \tag{7}$$

In the classical setting without the mean field, three components $\tilde{q}_s, \bar{\beta}_s, \bar{\alpha}_s$ of the optimal control are shown to exist in previous literature. The challenge we now solve is to show that given these three optimal components, we can construct the fourth component which is associated to the mean field operator.

Our main contribution in the second half consist of the study of the Mean-field reflected BSDE (MFRBSDE) and its connection to the optimal stopping problem of the dynamic Mean-field risk measure. To be more precise, we have showed the the existence/uniqueness and (strict) comparison results for the following reflected Mean-field BSDE corresponding to the counterpart Mean-field BSDE (5).

Mean-field Reflected BSDE is a process $(Y, Z, k(.), A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$ satisfying

$$\begin{cases} -dY_t = f(t, \omega, F(t, X_t(\cdot)), X_t, Z_t, k_t(\cdot))dt + dA_t - Z_t dW_t - \int_U k_t(u)\tilde{N}(dt, du); \quad Y_T = \xi_T, \\ Y_t \geq \xi_t,\ 0 \leq t \leq T \text{ a.s.}, \\ A \text{ is a nondecreasing RCLL continous process with } A_0 = 0 \text{ and such that} \\ \int_0^T (Y_t - \xi_t)dA^c_t = 0 \text{ a.s. and } \ \Delta A^d_t = -\Delta Y_t\, \mathbf{1}_{\{Y_{t^-} = \xi_{t^-}\}} \quad \text{a.s.} \end{cases} \tag{8}$$

The result that relates the value function to the optimal stopping problem with the solution to reflected BSDEs was proved in [12] for the case of BSDE driven with jumps. We extend the results in the context of Mean-field BSDEs. i.e. the equation in (4) holds for X to the solution in (5) and Y to the solution in (8). We also generalized the stream of results in [12] to the mean-field driven case, including the optimization principle and robust optimal stopping problems. Finally we Sketch the corresponding results and proofs for the another type of Mean-field (R)BSDE.

4 Optimal inventory management and order book modeling

Order book presentation. Most of electronic markets follow the order book matching rule. In such exchanges, buyers and sellers send their orders to a continuous-time double auction system. Then, orders are matched according to the price and time priority. Every submitted order has a specific price and size and the order book is the collection of all submitted and unmatched orders. We assume that every agent can take four basic decisions:

- **Decision l**: insert buying or selling limit orders (i.e quotes) in a queue or in the spread (if exists);
- **Decision s**: stay on the order book to keep its strategic placement;
- **Decision c**: cancel existing limit orders to be agressive or to place in the spread;
- **Decision m**: send market orders to get an instantaneous execution.

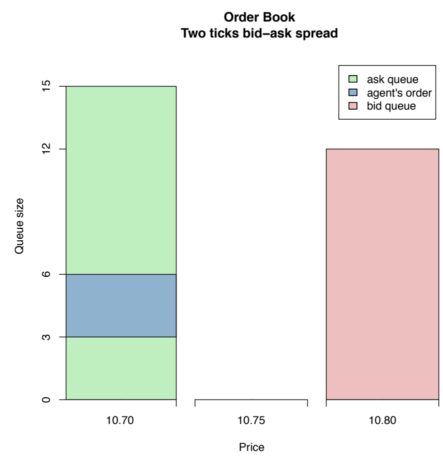


Figure 1: Idealized representation of the order book at a specific time

Moreover, more complex agent's decisions are combinations of the above four basic decisions.

Motivation. In this chapter, our main objective is to model the order book dynamics and the strategy of the market maker. Recently, the widespread market electronification facilitates the access of high quality data describing market participants decisions and interactions at the finest time scale. Furthermore, the availability of the order book data allows a better understanding of the market activity. Additionally, recent market fragmentation and the increase of trading frequency rise the complexity of agents actions and interactions. Thus, the comprehension of the order book dynamic has become a fundamental issue for all market participants and recent regulations emerge to increase the market transparency. Indeed, a deep understanding of the order book dynamic and agents behaviors enables: first, market makers to ensure liquidity provision at cheaper prices; second, HFT to reduce arbitrage opportunities; third, investors to reduce their transaction costs; fourth, policy makers to design relevant rules, to strengthen market transparency and to reduce market manipulation. Moreover, modeling the order book provides insights on the behavior of the price at larger time scales since the price formation process starts at the order book level. In Markov order book models, authors show, in [17], that the rescaled price process converges towards a Brownian motion. Hence, pricing models where the asset price follows a Brownian diffusion seem to be consistent with the order book mechanism.

Control problem. Since the market activity rises from the interaction between different agents acting optimally, our methodology consists in computing the optimal strategy of each agent and then use these optimal strategies to simulate the market. There are mainly three types of agents: market maker (MM), high frequency trading (HFT) firms and an institutional broker (IB). We will mainly consider the strategies for marker makers and use the simple strategies for the other two types of players. We express our control problem for order-driven markets (for example the equity market) since we may have multiple liquidity providers (i.e market makers). However, it can be easily adapted to the so-called quote-driven markets, where there are few monopolists liquidity provider who set the bid and ask quotes.

The MM are financial agents whose main job is to provide liquidity to other market participants. They can be designated by the exchange to ensure a basic level of liquidity provision. They propose liquidity mainly by inserting buying and selling limit orders (i.e quotes), and win potentially the bid-ask spread, in return. The MM may also send market orders to force the execution. Moreover, the MM need a fast access to the market to ensure attractive quotes and to reduce the adverse selection risk (i.e avoid to buy just before a price decrease). We give a detailed definition of the MM control problem in Section VIII.

The HFT agent main objective is to identify exploitable profit opportunities to catch or at the opposite loss situations to avoid. To this extent, the HFT agent use first sophisticated searching methodologies to identify profit/loss situations. Their searching methodologies are based on endogenous information associated to the order book state but also on exogenous information. In our case, for simplicity, the exogenous information is given by the dynamic of another asset highly correlated to the traded one. Then, the HFT agent need to react, to a market change, as fast as possible to catch the opportunity before it disappear. The HFT agent may provide liquidity using limit orders but they can also consume it via market orders. We give a detailed definition of the HFT control problem in Section 1.

The IB receives and executes investors meta-orders. A meta-order refers to a large investor buy or sell order which is executed in a succession of smaller orders. The IB execution algorithm consists in optimizing the execution price (i.e liquidation or acquisition price) of the meta-order given a benchmark price. In our case, we consider the VWAP price as benchmark. Since the execution time of the meta-order may last few hours, the IB execution algorithm time scale is coarser than MM and HFT reaction time scale. The IB use mainly market orders since the price impact cost of the meta-order (i.e fast execution of large quantities consumes order book liquidity and strongly moves the mid price) is higher than the bid-ask spread. We give a detailed definition of the IB control problem in Section 2.

Order book model. We model the order book state by a Markovian jump process, $U_t = (P_t, Q_t)$, where $P_t = (P_t^{\mathfrak{b}}, P_t^{\mathfrak{a}})$ and $Q_t = (Q_t^{\mathfrak{b}}, Q_t^{\mathfrak{a}})$. Here, $P_t^{\mathfrak{b}}$ and $P_t^{\mathfrak{a}}$ denote respectively the best available buying and selling prices, also called the best bid/ask prices. In this model, limit prices jump by a fixed size δ, representing the tick size, in such way that the spread $S := P_t^{\mathfrak{b}} - P_t^{\mathfrak{a}}$ takes values in $\{\delta, 2\delta\}$. Moreover, $Q_t^{\mathfrak{b}}$ (resp. $Q_t^{\mathfrak{a}}$) are the quantity offered at the best bid (resp. ask). When one limit is totally consumed, the order book is regenerated according a probability distribution $R^j(P_{t-}, Q_{t-}; \cdot)$ on $\mathbb{R} \times \mathbb{N}$ and $j \in \{\mathfrak{b}, \mathfrak{a}\}$. The kernels $R^j(P_{t-}, Q_{t-}; \cdot)$ with $j \in \{\mathfrak{b}, \mathfrak{a}\}$ depend on the depleted queue but also on the order book state (P_{t-}, Q_{t-}) before the regeneration. Moreover, we consider the following setting: the bid price can only move down by δ when the bid queue is depleted; the ask price can only increase by δ when the bid queue is depleted. We give a detailed description of the order book model in Section VIII.

Related literature. There are, essentially, two order book modeling approaches in the literature. First, the "general equilibrium models", including economists models, where the market activity is generated by interactions between rational agents who take optimal decisions (see [22, 37, 38]). Second, "statistical models" where the order book is seen as a random process (see [2, 3, 11, 16, 18, 19, 25, 33, 34, 39]). The statistical models focus on reproducing many salient features of real market rather than agents behaviors and interactions. In this paper, we study a general equilibrium model.

Whatever the modeling approach is, we need some basic assumptions for the order book state's dynamic. We differentiate essentially two types of state's dynamic depending on the time scale. First, the "order book resilience models" are adapted to the finest time scale since they provide details on the price and liquidity behavior [5, 13, 15, 26, 36]. Second, "black box models" view the order book from a coarser time scale since only the price dynamic is modeled [6, 14]. The order book resilience framework model the impact by a liquidity game: first a trade moves the price by the mechanical liquidity consumption, before the order book reacts by re-filling again (this is called resilience). Moreover, the order book resilience models are complex (see a sophisticated order book model in [27] and Hawkes based process in [5]) and difficult to calibrate while black box models are easy to calibrate since they use a small set of parameters. In this chapter, we consider an order book resilience for the MM and HFT control problem since they can react to every order book move. However, the IB state's dynamic follows a black box model since he is more sensitive to the price impact of the meta-order (i.e long term price move) than the quick moves in the order book.

In most of the order book resilience models, the arrival and cancellation follow independent Poisson processes. Such assumptions are not completely compatible with empirical evidences. In [2, 18, 21, 33, 40], authors show that its simplicity allows for the derivation of interesting formulas that can be tested on market data. To take into account the local behavior of the order book, in [29, 30], the authors get rid of the Poisson assumption, and present an order book liquidity model where order flows follow a Markovian jump process. They also provide ergodicity conditions and model parameters calibration methodology. This approach has been extended in two directions: model more consistent with market data and compatible with a control framework. To do this, they assume that insertion and cancellation intensities depend on both the size of Q^{Bid} and Q^{Ask}. Additionally, they model only the best limits, and focus on the regeneration process. Indeed, when one limit is totally consumed, the order book is regenerated in a new state whose regeneration law depends on the order book state just before the regeneration. This methodology reduces the dimension of the state process and enables high flexibility since the resurrection law depends on the killing state. In this paper, the state process is similar to the one of [1, 27], except that the spread value is no more constant; that is, in our setting, the spread takes values $\in \{\delta, 2\delta\}$. The dimension of the control space is also higher than [1, 27] since more complex decisions are required to tackle the market making problem.

In practice, optimal trading strategies are needed to find a trade-off between, at least, three factors: the price variation uncertainty, the market impact generated by market orders and limit orders and finally the inventory risk. The trading control problem depends on agents categories and within each category we may find slight differences as well, such as the risk aversion profile. However, the three above factors of risk are always involved. The MM control problem consists in handling at the same time buy and sell quotes to reduce the adverse selection risk (i.e avoid to buy just before a price decrease). Additionally, the MM handle their inventory exposed to price fluctuations mainly driven by the volatility of the market. In absence of price uncertainty, the MM can provide liquidity to an impatient buyer and wait the arrival of the next impatient seller with no risk. In the economics literature, the first market making strategies goes back to the eighties. In [28, 32], authors derives an optimal strategy for a monopolist dealer in a single stock. Subsequently, Grossman and Miller [23] study the risk faced by market makers and the equilibrium level of the liquidity. More recently, in [8, 14, 26], authors introduced the problem in the mathematical finance literature. They develop a market making strategy in the context of HFT using the limit order book. In our setting, the HFT control problem is similar to the MM one however the main difference lies in the used searching methodology: the HFT decisions are based on a more sophisticated private information. We take into account this information gap using pairs trading strategies. Finally, the IB main issue is to handle the permanent price impact generated by the execution of their meta-order. They also face an inventory risk associated to uncertain price fluctuations. The study of optimal liquidation deals started with the Almgren and Chriss framework using a mean-variance criterion. In [14, 24], the authors enhanced this approach to a more stochastic and liquidity driven framework. Finally, the optimal liquidation issue was studied in a more realistic situation using impulse control-driven strategies.

Contribution Our main contribution is to provide a comprehensive model for the order book dynamics and the optimal Market making problems. We present the equations satisfied by the MM optimal strategy and propose a numerical solution for this problem. Then we provide a methodology to simulate the market by where three types of players are interacted with each others.

5 Publications and Working Papers

- Chen R., Minca, A and Sulem A. (2017) Optimal connectivity for a large financial network. In ESAIM: Proceedings and Surveys, Vol. 59, p. 43-55.
- Chen R., Minca, A and Sulem A. Dynamic thresholds in a cascade model with application to systemic risk.
- Chen R., Dumitrescu R., Minca, A and Sulem A. Mean field BSDEs and global dynamic risk measures.

Part 2

Optimal connectivity for a large dynamic financial network subject to contagion risk

Chapter I

Network connectivity as equilibrium in a large game

In this chapter we use the structural approach to systemic risk. We formulate several models to investigate network connectivity for a set of financial institutions represented as nodes. Linkages are source of income, and at the same time they bear the risk of contagion. The optimal connectivity of the nodes results from a game, in which the risk of contagion depends on the choices of all nodes in the system. Our financial network model can be interpreted as a set of banks connected through funding relations, in which a node's threshold to contagion is represented by its external funding capacity. A second interpretation is that of a set of insurers connected through reinsurance contracts, in which the threshold to contagion is represented by their capital.

We begin by formulating a novel notion of equilibrium in a game with network interactions in the classical threshold contagion model. We then extend the threshold contagion model to the dynamic case by allowing recovery features. These allow various growth policies of the banks' assets between each round of contagion. Our results show that a higher heterogeneity in the initial distribution of the threshold (as captured by its standard deviation) implies a lower default probability in equilibrium, a higher heterogeneity in the initial distribution of the threshold also gives a larger average connectivity in equilibrium. Surprisingly, systems with higher growth/recovery rates may lead to equilibria with higher failure probability as well as higher final fraction of failed banks.

This chapter is organized as follows. In Section 1 we present the model of contagion and the nodes' optimization criteria. In section 2 we investigate the existence and uniqueness of the optimal connectivity, under an anticipated link failure probability. In Section 3 we analyze the equilibrium connectivity. Finally, in Section 4 we investigate numerically the equilibrium choice as a function of fundamentals.

1 Node performance under linkage benefits and contagion risk

We represent a financial system by n institutions (nodes), endowed with a sequence of thresholds $(\theta(i))_{i\in[n]}$. We can interpret the nodes as banks, in which case the thresholds represent the funding that these banks can raise from external lenders. Banks loan to other banks, and thus choose their connectivity and form funding linkages. Alternatively, we can interpret the nodes as insurance companies. Their threshold represents their capital. They can reinsure other insurance companies, and to do so they choose their connectivity and form reinsurance linkages.

Given a connectivity $\lambda(i),\ i \in [n]$, nodes form links according to the random matching from the configuration model. In the configuration model with given degree sequence $\lambda(i),\ i \in [n]$, each node i is assigned $\lambda(i)$ incoming half edges and $\lambda(i)$ outgoing half edges. The (multi)graph results from uniform matching of the in-coming half edges and the outgoing half edges. As $n \to \infty$, self loops and parallel edges become rare, and the graph is simple with positive probability [13]. This means that any property that holds with high probability on the configuration model, also holds with high probability conditional on this random graph being simple. This graph is denoted by $\mathcal{G}$ and we write $(i, j) \in \mathcal{G}$ for the event that there's a link between i and j. We let $\mu_n(\lambda, \theta)$ be the fraction of nodes with degree λ and threshold θ,

$$\mu_n(\lambda, \theta) := \frac{\#\{i \in [n] \mid \theta(i) = \theta,\ \lambda(i) = \lambda\}}{n}.$$

We assume the following regularity conditions $\mu_n(\lambda, \theta) \to \mu(\lambda, \theta)$, as n tends to ∞. We also assume the average connectivity converges to a finite limit

$$\sum_{\theta,\lambda} \lambda\mu_n(\theta, \lambda) \to \sum_{\theta,\lambda} \lambda\mu(\theta, \lambda) < \infty \tag{I.1}$$

We let the fraction of nodes with threshold θ, be defined as

$$\mu(\theta) := \sum_{\lambda} \mu(\lambda, \theta).$$

This network is subject to contagion risk. After the network is formed, a shock occurs and a set $\mathcal{D}_0$, representing a fraction ϵ of the entire system, switches its threshold to zero. If the financial network represents a network of funding relations among banks, the initial shock has the meaning that the debt capacity of certain banks becomes zero, i.e., no party lends to them, so they exogenously become illiquid. If the financial network represents a network of reinsurance contracts among insurance companies, then the initial shock is to be interpreted that the capital of certain insurers becomes zero following an extreme event. Under both of these representations, the thresholds becoming zero is equivalent to the failure of the node.

This initial set of failures triggers a cascade of failures, as we assume that failed nodes cut linkages from the nodes connected to them. Banks that fail after becoming illiquid cut loans from other banks in the system. These banks in turn also fail if their debt capacity is smaller than the amount of funding that was cut by failed banks. Insurers that fail after their capital becomes zero cut the reinsurance of other insurers. These other insurers in turn also fail if their capital is smaller than the amount of reinsurance that was cut by failed insurers. In sum, whenever a the node's threshold is smaller than the number of linkages that are cut from it, then it fails due to contagion. During the cascade processes, there is also a recovery feature in the whole financial system. This feature is captured by introducing the global growth rate (per unit time) $\alpha \cdot n$ of threshold for the system with n nodes, where $0 \leq \alpha < 1$. If the threshold is interpreted as funding to the banks from external lenders, then $\alpha \cdot n$ is the rate of growth per unit time of this funding to entire financial system. Similarly, if the threshold is interpreted as capital, then $\alpha \cdot n$ is the rate of growth of the entire capitalization of the financial system.

The link between this global growth and the individual growth of the threshold depends on how it is distributed among the banks. We assume the growth is distributed proportionally to the bank's number of links. That is, nodes with connectivity λ will have a growth with rate (per unit time) equal to $\frac{\alpha\cdot\lambda\cdot n}{\sum_{\theta,\lambda}\lambda\mu_n(\theta,\lambda)n}$ of their thresholds, where $\sum_{\theta,\lambda}\lambda\mu_n(\theta,\lambda)n$ gives the total number of links in the system. We further write this growth as $\frac{\alpha\cdot\lambda}{\bar{\lambda}_n}$, where $\bar{\lambda}_n = \sum_{\theta,\lambda}\lambda\mu_n(\theta,\lambda)$ represents the average connectivity in the system.

A very important feature is that only surviving banks can gain the growth while the failed ones remain defaulted, they will not benefit from the growth.

We now introduce a dynamic model of contagion. At time 0 banks in $\mathcal{D}_0$ are in default. Each of these banks cuts links to other banks. Each cut link represents an interaction and the number of interactions is always lower than the total number of linkages in the network $(n, \mathcal{G})$. In the dynamic model, we introduce the calendar time and relate it to interaction time. We will study the scaling limit of contagion size and we assume that the total (calendar) time for all interactions is independent of n. Since the number of links scales linearly with n (see (I.1)) then the time between interactions must scale with $\frac{1}{n}$. For a financial system with n nodes, we define T_k^n the calendar time of the k^{th} interaction and we refer to k to the interaction time.

We assume that the duration in calendar time between the two successive interactions follows an exponential distribution with parameter n, i.e. $\Delta_k = T_k^n - T_{k-1}^n \sim Exp(n)$. This instantaneous reward mechanism in the dynamic case allows the threshold to grow $\frac{\alpha\cdot\lambda}{\bar{\lambda}_n}\ \Delta_k$ between the two interactions.

The dynamics of interactions is as follows: links that belong to failed banks are revealed one by one (initially all such links are unrevealed). After each exponential time, a link belonging to a failed bank is revealed[1] and the survival condition of the counterparty node is checked according to its current threshold. If the defaulted link of the counterparty exceeds its current threshold, the node defaults and its links become unrevealed failed links. The cascade progresses until there are no more unrevealed failed links. Therefore it stops at most after $\bar{\lambda}_n n$ interactions. We let $\mathcal{D}_f$ the set of failures at the end of the contagion process.

We have the following theorem, which characterizes the set of failures in the case $\alpha = 0$.

Theorem 1.1 ([1]). *Let p^* be the smallest fixed point of the map J in $[0,1]$, where*

$$J(p) := \sum_{\theta,\lambda} \frac{\lambda\mu(\theta,\lambda)}{\sum_{\theta,\lambda}\lambda\mu(\theta,\lambda)} \cdot B(\theta,\lambda,p),$$

$$B(\theta,\lambda,p) := \mathbb{P}(Bin(\lambda,p) \geq \theta) = \sum_{l\geq\theta}^{\lambda} \binom{\lambda}{l} p^l (1-p)^{\lambda-l}.$$

Here $Bin(\lambda,p)$ denotes a random variable with binomial distribution with parameters λ and p.

(i) *If $p^* = 1$, i.e., if $J(p) > p$ for all $p \in [0,1)$, then asymptotically (when $n \to \infty$) almost all nodes fail during the cascade.*

(ii) *If $p^* < 1$ and p^* is a stable fixed point of J, i.e., $J'(p^*) < 1$, then the final fraction of failures converges in probability to*

$$\sum_{\lambda,\theta} \mu(\lambda,\theta) \sum_{\theta} B(\theta,\lambda,p^*). \tag{I.2}$$

[1] The choice is uniform among all unrevealed links belonging to failed banks.

The asymptotic fraction of surviving nodes with degree λ and threshold θ at the end of the contagion is given (in the limit) by

$$s^{\theta,\lambda} = \mu(\theta,\lambda)(1 - B(\theta,\lambda,p^*)).$$

Theorem 1.1 states that in the asymptotic limit the fraction of defaults can be described by means of the binomial distribution: given the probability p^* that a link is failed, then the probability of failure of a node with connectivity λ and threshold θ is approximated by $B(\theta,\lambda,p^*)$, the probability of failure *as if* the links' default event were independent. For this reason we will refer to p^* as the global failure probability, which is the probability that a link chosen at random leads to a failed node at the end of the cascade, in the limit when $n \to \infty$. The fraction of defaults is given by (I.2). Note that a fixed point as in the theorem always exists as we check that $J(0) \geq 0$ and $J(1) \leq 1$ and the function J is continuous.

Case $\alpha > 0$. We can now extend Theorem 1 to the case where the global growth rate is $\alpha > 0$. We will prove that a similar approximation also holds in this case. The proof is more involved because the threshold to default at any point in time is no longer constant and equal to θ, but grows at a rate α. Nodes that fail during the contagion process will not benefit from this growth in the future, only surviving nodes can grow at any given time. This presents challenges in the description and the analysis of the system.

In the case without growth, it was sufficient at any time to keep track of the number of failed linkages since default happens when the number of failed linkages reaches the initial threshold. In contrast, here banks default at the first time when the number of failed links reaches the initial threshold plus the growth up to that time, so it is insufficient to keep track only of the current number of failed links. We need to keep track of cumulative failed links process (which is an increasing jump process with jump size one). If this process has ever crossed the threshold (with linear growth), then the bank has failed, so the failure of a bank is a first passage problem.

Remarkably, one can give a heuristic to compute the probability that a bank defaults based on a notion of average growth and the notion of global failure probability. The rigorous proof is given in Theorem 1.2.

Heuristic of node failure probability computation : We introduce $B^\alpha(\theta,\lambda,p^*)$ as the the failure probability of a node with degree λ, threshold θ when the growth rate is α and the global failure probability is p^*. We introduce $\beta^{\theta,\lambda,l}(p^*)$ as the probability that the nodes with initial threshold θ and connectivities λ survive under the global failure probability p^* and have l failed links at the end of cascade. The quantity $B^\alpha(\theta,\lambda,p^*)$ can be calculated by:

$$B^\alpha(\theta,\lambda,p^*) := 1 - \sum_{l \leq \min\{[\theta+\alpha\lambda p^*]-1,\lambda\}} \beta^{\theta,\lambda,l}(p^*).$$

A condition for the node to survive is that the number of failed links l ($\leq \lambda$) does not exceed the final threshold $\theta + \alpha\lambda p^*$ which is the initial threshold plus the growth gain. The growth gain is proportional to λ and to p^*. This is because the global failure probability p^* can also be thought of as the duration of the contagion: the higher the global failure probability, the longer the contagion lasts. In turn, if the cascade lasts for longer then banks that survive have also received growth benefits for longer.

We can check that $B^{\alpha}(0, \lambda, p^*) = 1$, i.e., a bank whose threshold is zero is in default. This follows by definition, since $\beta^{0,\lambda,l}(p^*) = 0$. Moreover, we check that $B^{\alpha}(\theta, \lambda, 0) = 0$ if $\theta > 0$ (all banks survive). This is due to

(i) $l = 0$, then $\beta^{\theta,\lambda,l}(0) := \binom{\lambda}{0}0^0(1-0)^{\lambda} = 1$

(ii) $0 < l \leq \theta$, then $\beta^{\theta,\lambda,l}(0) := \binom{\lambda}{l}0^l(1-0)^{\lambda-l} = 0$

(iii) $l > \theta$, then $\beta^{\theta,\lambda,l}(p) = 0$; In general, when $0 \leq p \leq \frac{l-\theta}{\alpha\lambda}$, we have $\beta^{\theta,\lambda,l}(p) = 0$ (by Theorem1.2 (ii)).

These gives the second term sum equals to 1.

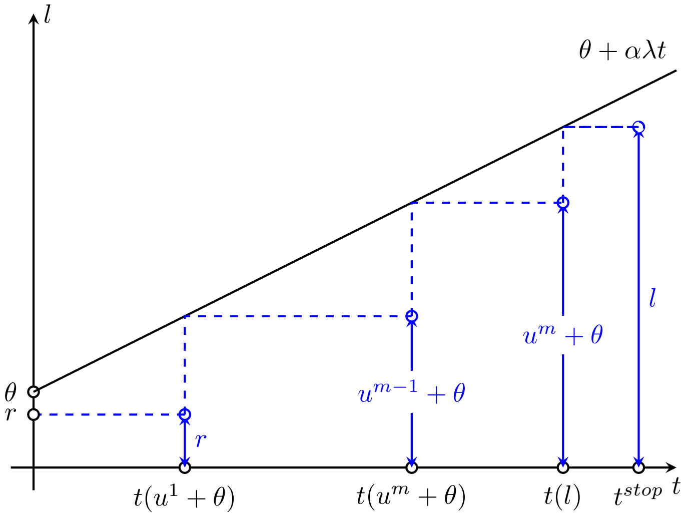
We now proceed to give the heuristic for the computation of β. For the case when the number of failed links l is smaller or equal than the initial threshold θ, then the survival probability $\beta^{\theta,\lambda,l}(p^*)$ is simply the binomial distribution (the probability to have l failed links)

$$\beta^{\theta,\lambda,l}(p^*) = \binom{\lambda}{l}(p^*)^l(1-p^*)^{\lambda-l},$$

since each link is exposed to failure probability p^*.
For the case when l is larger than the initial threshold θ, the actual calculation of $\beta^{\theta,\lambda,l}(p^*)$ is more involved.

If the number of failed links is $l > \theta$ then the bank is definitely failed if the growth cannot cover $l - \theta$, i.e. if $\alpha p^* \lambda < l - \theta$, which gives $\beta^{\theta,\lambda,l}(p) = 0$ for $0 \leq p \leq \frac{l-\theta}{\alpha\lambda}$. In contrast, if $p > \frac{l-\theta}{\alpha\lambda}$ then we need to make sure that the bank survives. Namely, it needs to satisfy the survival condition at the end of contagion and also at each time before. This makes the computation more involved. Remarkably, the solution has a combinatorial representation, that we show in Figure 1. The key point is to identify the critical times when the threshold process could be crossed by the cumulative failed links process. Let t the current time in the spread of contagion. As we will show, the cascade will end at a time t^{stop} when all failed linkages have been explored and is related to the global link failure probability p^* by a scaling constant: $\frac{t^{stop}}{\lambda} = p^*$. The longer the contagion lasts the larger the global link failure probability. We note that the threshold process for a node with initial threshold θ and connectivity λ is $\theta + \alpha\lambda t$, $t \in [0, t^{stop}]$. Recall that we are computing the survival probability of such a node, *given* that its final number of failed linkages is l. For every $u \in [\theta + 1, l]$ we let $t(u) := \frac{u-\theta}{\alpha\lambda}$ the time when the bank's threshold process is equal to u and thus the bank can withstand up to u failed links at this time. In order to ensure that the bank survives, we need to check that the number of failed links at time $t(l)$ is lower than l. The first case is when this number is $r \leq \theta$. In this case, we are sure that the cumulative failed links process has not crossed the threshold process at any time $s \in [0, t(l)]$. Therefore the survival probability in this case is simply the probability that there are r failed linkages between 0 and $t(l)$ times the probability that there are $l - r$ failed linkages between the time $t(l)$ and t^*. The proof of Theorem 1.2 suggests that the default probability of a linkage in any time interval is proportional to the length of that interval (with the scaling $\frac{1}{\lambda}$).

The second case is when the number of failed links at time $t(l)$ is $\theta + u^m$, for some $u^m \in [1, l-\theta-1]$. Then we have a backward recursion by which we determine previous times when we need to check that the crossing has not happened. The previous time when such crossing could have happened is $t(\theta+u^m)$, since between $t(\theta+u^m)$ and $t(l)$, the threshold processes is definitely above $\theta+u^m$. Thus we only need to check that the number of failed links at time $t(\theta+u^m)$ is given by $\theta+u^{m-1}$ for a $u^{m-1}<u^m$. By the same reasoning, we need to check the at time $t(\theta+u^{m-1})$ the number of failed links is given by $\theta+u^{m-2}$ for a $u^{m-2}<u^{m-1}$ and so on, until at time $t(\theta+u^1)$ we need to check that the number of failed links is $r\leq\theta$. Then the survival probability is the product of the probabilities that there are r, $\theta+u^1-r \ldots$, $u^{m-1}-u^{m-2}$, u^m-u^{m-1} and $l-\theta-u^m$ in the respective time intervals $[0,t(\theta+u^1)]$, $[t(\theta+u^1),t(\theta+u^2)]$ $\ldots$, $[t(\theta+u^{m-1}),t(\theta+u^m)]$, $[t(\theta+u^m),t(l)]$ and finally $[t(l),t^{stop}]$. It is understood that m is a discrete variable which takes values in $[1,l-1-\theta]$ (this is the number of times it would be possible to cross the threshold process).



With the intuition behind the survival probability computation, we can now turn to finding the global link default probability p^*. Our theorem below shows this quantity is the solution to the fixed point equation $p^*=J(p^*)$, where the function J makes use of the survival probability B^α.

To see that this fixed point represents the global link default probability, let us multiply both sides of the fixed point equation with $n\sum_{\theta,\lambda}\lambda\mu(\theta,\lambda)$, which represents the total number of links in the network. Then on the lefthand side we have $p^*n\sum_{\theta,\lambda}\lambda\mu(\theta,\lambda)$ which represents the expected total number of failed links present in the system at the end of the cascade. The number of failed links at the end of the cascade can be also accounted as the sum of expected failed links across different nodes. Indeed, $\lambda n\mu(\theta,\lambda)B^\alpha(\theta,\lambda,p^*)$ gives the expected number of failed links from nodes with threshold θ and connectivities λ, since $B^\alpha(\theta,\lambda,p^*)$ represents the failure probability of node with threshold θ and connectivities λ while $\lambda n\mu(\theta,\lambda)$ counts the total number of links belongs to such nodes. Summing up over θ and λ we have that $\sum_{\theta,\lambda}\lambda n\mu(\theta,\lambda)B^\alpha(\theta,\lambda,p^*)$ also gives the total number of failed links. The fixed point equation $p^*=J(p^*)$ states that the second way to account for failed links reaches the same value as the first one.

Theorem 1.2. *Let p^* be the a relaxed fixed point defined as $p^*:=\min\{p,J(p)\leq p,p\in[0,1)\}$,* *where*

$$J(p) := \sum_{\theta,\lambda} \frac{\lambda\mu(\theta,\lambda)}{\sum_{\theta,\lambda}\lambda\mu(\theta,\lambda)} \cdot B^{\alpha}(\theta,\lambda,p),$$

$$B^{\alpha}(\theta,\lambda,p) := 1 - \sum_{l\le\min\{[\theta+\alpha\lambda p],\lambda\}} \beta^{\theta,\lambda,l}(p)$$

where for fixed (θ,λ,p)*,* $(\beta^{\theta,\lambda,l}(p))_{l\ge 0}$ *is defined as follows*

(i) $l \le \theta$*, then* $\beta^{\theta,\lambda,l}(p) := \binom{\lambda}{l} p^l (1-p)^{\lambda-l}$.

(ii) $l > \theta$ *and* $0 \le p \le \frac{l-\theta}{\alpha\lambda}$ *then* $\beta^{\theta,\lambda,l}(p) = 0$.

(iii) $l > \theta$ *and* $p > \frac{l-\theta}{\alpha\lambda}$*, then*

$$\begin{aligned}\beta^{\theta,\lambda,l}(p) = \Bigg[&\sum_{r\le\theta} \binom{\lambda-r}{l-r}(1-p+\frac{l-\theta}{\alpha\lambda})^{\lambda-l}(p-\frac{l-\theta}{\alpha\lambda})^{l-r}\binom{\lambda}{r}(1-\frac{l-\theta}{\alpha\lambda})^{\lambda-r}(\frac{l-\theta}{\alpha\lambda})^{r} \\ &+ \sum_{1\le m\le l-\theta-1}\ \sum_{1\le u^1<\ldots u^m<l-\theta-1}\ \sum_{r\le\theta} \binom{\lambda-\theta-u^m}{l-\theta-u^m}(1-p+\frac{l-\theta}{\alpha\lambda})^{\lambda-l}(p-\frac{l-\theta}{\alpha\lambda})^{l-\theta-u^m} \\ &\binom{\lambda-\theta-u^{m-1}}{\theta+u^m-\theta-u^{m-1}}(1-\frac{l-\theta-u^m}{\alpha\lambda})^{\lambda-\theta-u^m}(\frac{l-\theta-u^m}{\alpha\lambda})^{\theta+u^m-\theta-u^{m-1}}\cdots \\ &\binom{\lambda-u^1-\theta}{\theta+u^2-\theta-u^1}(1-\frac{u^3-u^2}{\alpha\lambda})^{\lambda-\theta-u^2}(\frac{u^3-u^2}{\alpha\lambda})^{\theta+u^2-\theta-u^1} \\ &\binom{\lambda-r}{\theta+u^1-r}(1-\frac{u^2-u^1}{\alpha\lambda})^{\lambda-\theta-u^1}(\frac{u^2-u^1}{\alpha\lambda})^{\theta+u^1-r}\binom{\lambda}{r}(1-\frac{u^1}{\alpha\lambda})^{\lambda-r}(\frac{u^1}{\alpha\lambda})^{r}\Bigg].\end{aligned} \tag{I.3}$$

We have

1. *If* $p^* = 1$*, i.e., if* $J(p) > p$ *for all* $p \in [0,1)$*, then asymptotically (when* $n \to \infty$*) almost all nodes fail during the cascade.*

2. *If* $p^* < 1$ *and* p^* *is a stable fixed point of* J*, i.e.,* $J'(p^*) < 1$*, then the final fraction of failures converges in probability to* $\sum_{\lambda,\theta}\mu(\lambda,\theta)\sum_{\theta} B^{\alpha}(\theta,\lambda,p^*)$.

The asymptotic fraction of surviving nodes with degree λ *and threshold* θ *at the end of the contagion is given (in the limit) by*

$$s^{\theta,\lambda} = \mu(\theta,\lambda)(1 - B^{\alpha}(\theta,\lambda,p^*)).$$

Remark 1.1. *When* $\alpha = 0$*, we recover that* $B^0(\theta,\lambda,p) = B(\theta,\lambda,p)$.

Remark 1.2 (Existence of a fixed point)**.** *The relaxed fixed point always exists since* $J(0) \ge 0$ *and* $J(1) \le 1$*, even if* J *is discontinuous. When the function* J *is continuous on* $[0,1)$*, the relaxed fixed point is the same as the standard fixed point.*

This result is established by describing the contagion process using a Markov chain of lower dimension than the initial system, in which we aggregate nodes according to their connectivity, threshold, and number of failed counterparties. From the point of view of the evolution of the cascade, the nodes in the same class are indistinguishable. We then show that, as the network size increases, the Markov chain rescaled by network size converges in probability to a limit described by a system of ordinary differential equations, which can be solved in closed form. This readily gives us the asymptotic fraction of surviving nodes (in each class of connectivity and threshold) at each time of the cascade spread, namely $s^{\theta,j}(t)$. The stopping time of the cascade t^{stop} is the first time when there are no more unexplored failed linkages. We can relate the stopping time of the cascade to the global failure risk captured by p^* and in the sequel we will use this quantity to define the nodes' performance criteria and define their connectivity optimization problem.

Remark 1.3. *Note that the ability to define a global failure probability depends on our choice of the configuration model as an underlying network. The configuration model is a flexible setting as it gives the possibility to use any degree distribution, but it carries the implicit assumption that a linkage default probability is independent of the end node in equilibrium. It remains an open problem to extend the equilibrium analysis below to other types of random graphs, e.g. inhomogeneous random graphs, and for case when the global failure probability is replaced by a failure probability matrix (with the default probability of a link depending on the types of its end nodes).*

Banks' basic tradeoff is the following: as they add more connectivity, they increase the risk of contagion. At the same time they derive more benefits from their linkages. We capture these benefits in a simple way, by assuming that there exists a numéraire and that surviving nodes with connectivity λ receive λ (units of the numéraire) at the end of the cascade. They receive no benefit from their linkages if they fail.

We are now ready to define the nodes' performance measure.

Definition 1.4 (Nodes' performance)**.** *We define the performance criterion for a node with degree λ and threshold θ as the expected benefit of linkages, and it is given in the asymptotic limit by*

$$\lambda\big(1 - B^\alpha(\theta, \lambda, p^*)\big),$$

with p^* given in Theorem 1.2.

Note that the global failure probability of a link p^* depends on the connectivity choice of all nodes. Therefore, the optimal connectivity is an outcome of an equilibrium. We proceed in two steps to determine this equilibrium. In the first step, we let nodes choose their connectivity according to an expected failure probability p of a link, i.e., a node with threshold θ chooses a connectivity $\lambda(\theta, p)$

$$\lambda^*(\theta, p) \in \arg\max_\lambda \lambda\big(1 - B^\alpha(\theta, \lambda, p)\big). \tag{I.4}$$

In the second step, detailed in Section 3, we will impose an equilibrium condition that the expected failure probability of a link coincides with the actual failure probability of a link, under the optimal connectivity.

Before we proceed, we note that the banks maximize an expected payoff which is equal to the connectivity if the bank doesn't fail and 0 if it does. One could argue that in the performance criterion banks should only count the profit from the surviving linkages. In fact, whether we should use the expected benefit from all linkages or the expected benefit from the surviving links depends on the model interpretation. It can be of interest to use the expected benefit and not the expected benefit from the surviving links when surviving banks replace the defaulted links with new funding from from the outside creditors to maintain their business. In such case the replaced linkages have the same function as the original ones.

One could also assume that failed links do not provide any benefit and maximize the product of the probability of survival $1 - B^\alpha(\theta, \lambda, p)$ and the expectation of non-failed links $\lambda(1-p)$. The next remark shows that for fixed failure probability p of a link, the banks' choice remains the same. This in turn implies the equilibrium and its stability properties remain the same.

Remark 1.5 (Robustness to the performance criterion). *For fixed θ and p, we let*

- $V_1(\lambda) := (1-p)\lambda\big(1 - B^\alpha(\theta, \lambda, p)\big)$. *In this case the optimizer remains the same as above since*
$$\max_\lambda V_1(\lambda) = (1-p)\max_\lambda \lambda\big(1 - B^\alpha(\theta, \lambda, p)\big).$$
Thus
$$\arg\max_\lambda V_1(\lambda) = \arg\max_\lambda \lambda\big(1 - B^\alpha(\theta, \lambda, p)\big).$$

When the direction of link is the same as the direction of contagion (i.e. i loans to j and there is contagion from i to j and the link (i, j) fails when i withdraws funding from j) then V_1 represents the expected benefit of the non-failed links of the surviving banks.

2 Nodes' optimal connectivity choice

In this section we investigate existence and uniqueness of the node's optimal connectivity.

2.1 The static case ; No recovery feature ($\alpha = 0$)

This case of zero growth is investigated in [7]. We begin by noting that the node's performance criterion balances increased benefits from connectivity (via the factor λ) against increased risk, via the decreasing surviving probability $1 - B(\theta, \lambda, p)$.

Proposition 2.1 (Existence). *When $\alpha = 0$, the optimization problem (I.4) admits a finite optimizer $\lambda^*(\theta, p)$ which lies in the interval $\left[\frac{1-p}{p} \vee (\theta - 1), \frac{\theta - 2p + \sqrt{(\theta-2p)^2 + 4(1-p)p}}{2p}\right]$. This interval is non-empty when $\theta \geq 1$. Moreover, the optimal connectivity lies in $\left[\frac{1-p}{p} \vee (\theta-1), \frac{\theta}{p} - 1\right]$.*

Proof.

We will show that the node's performance $V(\lambda)$ is decreasing in λ for $\lambda > \frac{\theta - 2p + \sqrt{(\theta-2p)^2 + 4(1-p)p}}{2p}$ and is increasing for $\lambda < \frac{1-p}{p} \vee \theta$.

Recall that $\binom{\lambda+1}{l} = \frac{\lambda+1}{\lambda+1-l}\binom{\lambda}{l} = \frac{1}{1-l/(\lambda+1)}\binom{\lambda}{l}$. We have

$$V(\lambda+1) - V(\lambda) = \sum_{l\le\theta-1} [(\lambda+1)\frac{1-p}{1-l/(\lambda+1)} - \lambda]\binom{\lambda}{l}(1-p)^{\lambda-l}p^l.$$

Now if $\lambda > \frac{-(2-\theta)+2(1-p)+\sqrt{(2-\theta-2+2p)^2+4(1-p)p}}{2p} = \frac{\theta-2p+\sqrt{(\theta-2p)^2+4(1-p)p}}{2p}$, then we have $(\lambda+1)\frac{1-p}{1-(\theta-1)/(\lambda+1)} - \lambda < 0$. This gives $(\lambda+1)\frac{1-p}{1-l/(\lambda+1)} - \lambda < 0$, for all $l \le \theta - 1$ since $(\lambda+1)\frac{1-p}{1-l/(\lambda+1)} - \lambda$ is increasing in l. This leads to $(\lambda+1)^2(1-p) < \lambda(\lambda+2-\theta)$. Thus $\lambda > \frac{-(2-\theta)+2(1-p)+\sqrt{(2-\theta-2+2p)^2+4(1-p)p}}{2p} = \frac{\theta-2p+\sqrt{(\theta-2p)^2+4(1-p)p}}{2p}$. Therefore if $\lambda > \frac{\theta-2p+\sqrt{(\theta-2p)^2+4(1-p)p}}{2p}$, then $V(\lambda+1) - V(\lambda) < 0$. Thus the optimal λ^* should lie in the interval $[0, \frac{\theta-2p+\sqrt{(\theta-2p)^2+4(1-p)p}}{2p}]$.

Similarly if $\lambda < \frac{1-p}{p}$, then $(\lambda+1)(1-p) - \lambda > 0$ and thus we have $(\lambda+1)\frac{1-p}{1-l/(\lambda+1)} - \lambda > 0$ for all $l \le \theta - 1$, again by the monotonicity in l. This gives that $V(\lambda+1) - V(\lambda) > 0$ which means the optimal λ^* should be greater than $\frac{1-p}{p}$. Also it is obvious to choose $\lambda^* \ge \theta$ for players with threshold θ. We conclude that the optimal λ^* should lie in the interval $\left[\frac{1-p}{p} \vee \theta, \frac{\theta-2p+\sqrt{(\theta-2p)^2+4(1-p)p}}{2p}\right]$.

Remark 2.2 (Robustness to the performance criterion when $\alpha = 0$)**.** *Apart from the performance criterion V_1 defined in Remark 1.5, we can also define another performance criterion when $\alpha = 0$. Namely, for fixed θ and p,*

- $V_2(\lambda) := \sum_{l<\theta}(\lambda - l)\binom{\lambda}{l}p^l(1-p)^{\lambda-l}$.
  Straightforward computation shows that

$$V_2(\lambda) = \sum_{l<\theta} \lambda\binom{\lambda-1}{l}p^l(1-p)^{\lambda-l} = (1-p)\lambda\big(1 - B(\theta, \lambda-1, p)\big).$$

  Thus

$$\arg\max_{\lambda} V_2(\lambda) = \arg\max_{\lambda} \lambda\big(1 - B(\theta, \lambda-1, p)\big).$$

  which shares the same structure as the original criterion and will lead to similar results.

When the direction of the link is inverse to the direction of contagion (i.e. i loans to j and there is contagion from j to i and the link (i, j) fails because j is insolvent or cannot pay the loan) then V_2 represents the expected benefit of the non-failed links of the surviving banks.

Remark 2.3. *The same results hold for the performance criterion V_2. We mirror the proof of the existence result, Proposition 2.1. We directly obtain that the optimizer λ^* lies in the interval $\left[\frac{1-p}{p} \vee \theta, \frac{\theta}{p} - 1\right]$.*

We will show that the node's performance $V_2(\lambda)$ is decreasing in λ for $\lambda > \frac{\theta-2p+\sqrt{(\theta-2p)^2+4(1-p)p}}{2p}$ and is increasing for $\lambda < \frac{1-p}{p} \vee \theta$. Recall that $\binom{\lambda}{l} = \frac{\lambda}{\lambda-l}\binom{\lambda-1}{l} =$

$\frac{1}{1-l/(\lambda)}\binom{\lambda-1}{l}$. We have

$$V_2(\lambda+1)-V_2(\lambda)=\sum_{l\leq\theta-1}[(\lambda+1)\frac{1-p}{1-l/\lambda}-\lambda]\binom{\lambda-1}{l}(1-p)^{\lambda-1-l}p^l.$$

Now if $\lambda > \frac{\theta-p}{p}$, then we have $(\lambda+1)\frac{1-p}{1-(\theta-1)/\lambda}-\lambda<0$. This gives $(\lambda+1)\frac{1-p}{1-l/\lambda}-\lambda<0$, for all $l\leq\theta-1$ since $(\lambda+1)\frac{1-p}{1-l/\lambda}-\lambda$ is increasing in l. This leads to $(\lambda+1)(1-p)<\lambda+1-\theta$. Thus $\lambda>\frac{\theta-p}{p}$. Therefore if $\lambda>\frac{\theta-p}{p}$, then $V_2(\lambda+1)-V_2(\lambda)<0$. Thus the optimal λ^* should lie in the interval $[0,\frac{\theta-p}{p}]$.

Similarly if $\lambda<\frac{1-p}{p}$, then $(\lambda+1)(1-p)-\lambda>0$ and thus we have $(\lambda+1)\frac{1-p}{1-l/\lambda}-\lambda>0$ for all $l\leq\theta-1$, again by the monotonicity in l. This gives that $V_2(\lambda+1)-V_2(\lambda)>0$ which means the optimal λ^* should be greater than $\frac{1-p}{p}$. Also it is obvious to choose $\lambda^*\geq\theta$ for players with threshold θ. We conclude that the optimal λ^* lies in the interval $\left[\frac{1-p}{p}\vee\theta,\frac{\theta-p}{p}\right]$.

2.2 The dynamic threshold case ($\alpha > 0$)

We now consider the general case of a node's optimization problem for $\alpha>0$. It is understood that all other nodes' connectivities and the anticipated global failure probability are fixed. A finite optimizer exists, and as such it can be used to obtain the equilibrium across nodes in the next section.

Proposition 2.4 (Existence)**.** *The optimization problem (I.4) admits a finite optimizer $\lambda^*(\theta,p)$.*

Proof.

We define $V^\alpha(\lambda):=\lambda\big(1-B^\alpha(\theta,\lambda,p)\big)$ which is bounded by the following quantity:

$$U^\alpha(\lambda):=\lambda\cdot\left[\sum_{l\leq\theta+\alpha\lambda p}\binom{\lambda}{l}(1-p)^{\lambda-l}p^l\right]=\lambda\cdot I_{1-p}(\lambda-\theta+\alpha\lambda p,\theta+\alpha\lambda p+1)$$

We next recall the following estimates: if $k\leq np$, then

$$I_{1-p}(n-k,k+1)\leq exp(-\frac{(np-k)^2}{2pn}) \tag{I.5}$$

Since $\alpha<1$, there exist λ_0 such that when $\lambda>\lambda_0$, we have $\theta+\alpha\lambda p\leq\lambda p$.
This gives

$$V^\alpha(\lambda)\leq U^\alpha(\lambda)\leq\lambda\exp\left(-\frac{(\lambda p-\theta-\alpha\lambda p)^2}{2p\lambda}\right)=\lambda\exp[-\frac{(1-\alpha)^2}{2}\lambda p+\frac{(1-\alpha)}{2}\theta-\frac{\theta^2}{2\lambda p}]$$

The righthand side tends to 0 when $\lambda\to+\infty$. Thus the maximizer exists and is finite.

Remark 2.5. *The quantity U^α gives the value function for the same optimization problem for a modified system in which we allow defaulted banks to receive gains from their linkages and even recover, so it is intuitive that $V^\alpha(\lambda)\leq U^\alpha(\lambda)$. We formally check that $V^\alpha(\lambda)\leq U^\alpha(\lambda)$ by using the inequality $\beta^{\theta,\lambda,l}(p)\leq\binom{\lambda}{l}(1-p)^{\lambda-l}p^l$. (See Remark 6.2)*

3 Equilibrium

We now impose the equilibrium condition that the anticipated global failure probability failure probability coincides with the global failure probability given by Theorem 1.2 when nodes are at their optimal connectivity.

Definition 3.1. *We call* $(p^*, (\lambda^*(\theta), \theta \geq 0))$ *a standard equilibrium if*

- *Given* p^*
$$\lambda^*(\theta) \in \arg\max_{\lambda} \lambda\big(1 - B^\alpha(\theta, \lambda, p^*)\big), \quad \textit{for each } \theta; \tag{I.6}$$

- p^* *is the smallest solution of the fixed point equation :*
$$p^* = \sum_\theta \frac{\lambda^*(\theta)\mu(\theta)}{\sum_\theta \lambda^*(\theta)\mu(\theta)} \cdot B^\alpha(\theta, \lambda^*(\theta), p^*). \tag{I.7}$$

The fixed point equation in the equilibrium definition is derived from the fixed point equation in Theorem 1.2, where the connectivity is set to $\lambda^*(\theta)$ and we note that $\mu(\theta) = \mu(\theta, \lambda^*(\theta))$.

Combining the two conditions above, we see that in equilibrium the failure probability of a link p^* is the smallest fixed point of the map

$$\phi^\alpha(p) := \sum_\theta \frac{\lambda^*(\theta, p)\mu(\theta)}{\sum_\theta \lambda^*(\theta, p)\mu(\theta)} \cdot B^\alpha(\theta, \lambda^*(\theta, p), p), \tag{I.8}$$

where $\lambda^*(\theta, p)$ is defined in (I.4).

Now we study the existence of the equilibrium.

Proposition 3.2 (Existence of equilibrium)**.** *When* $\lambda^*(\theta, p)$ *is continuous in* $p \in [0, 1)$*, the function* ϕ^α *admits at least one fixed point* p^**. When* $\lambda^*(\theta, p)$ *is not continuous in* p*, the map* ϕ *admits a relaxed fixed point* p^* *defined as* $p^* := \min\{p, \phi^\alpha(p) \leq p,\ p \in [0, 1)\}$*. For both cases, we let* $\lambda^*(\theta) = \lambda^*(\theta, p^*)$*.*

Proof. The proof is immediate: when $\lambda^*(\theta, p)$ is continuous in $p \in [0, 1)$ then the function ϕ^α is continuous in $[0, 1)$. It thus admits at least one fixed point since $\lim_{p\to 1} \phi^\alpha(p) \leq \sum_\theta \mu(\theta) \frac{\lambda^*(\theta,p)}{\sum_\theta \lambda^*(\theta,p)\mu(\theta)} = 1$ and $\lim_{p\to 0} \phi^\alpha(0) \geq 0$. When $\lambda^*(\theta, p)$ is not continuous, the relaxed fixed point always exists since $\phi^\alpha(0) \geq 0$ and $\phi^\alpha(1) \leq 1$.

Proposition 3.3. *The continuity of the map* $p \to \lambda^*(\theta, p)$ *holds under uniqueness of the optimal connectivity* $\lambda^*(\theta, p)$*.*

Proof. Suppose uniqueness holds, for fixed θ, if $p_i \to p$, let $\lambda_i = \arg\max_\lambda \lambda\big(1 - B^\alpha(\theta, \lambda, p_i)\big)$, we may assume $\lambda_i \leq K$ for some K. Now suppose λ^* is an accumulation point of λ_i, that is, $\lambda_{(i)} \to \lambda$ over some subsequence. Since $\lambda\big(1 - B^\alpha(\theta, \lambda, p)\big)$ is continuous in λ and p, we have $F(p) := \max_\lambda \lambda\big(1 - B^\alpha(\theta, \lambda, p)\big)$ is continuous in p. Thus by definition $F(p_i) = \lambda_i\big(1 - B^\alpha(\theta, \lambda_i, p_i)\big)$. Taking limit of both side on subsequence we obtain $F(p) = \lambda\big(1 - B^\alpha(\theta, \lambda, p)\big)$ which gives $\lambda^* = \arg\max_\lambda \lambda\big(1 - B^\alpha(\theta, \lambda, p)\big)$. In particular, any accumulation point of the sequence p_i is a optimizer. Since the optimizer is unique, the only accumulation point of the sequence λ_i is the optimizer λ^*. Thus we have proved that any subsequence of λ_i has a further subsequence that converge to λ^*, which indicates $\lambda_i \to \lambda^*$. This gives $\lambda(p_i) \to \lambda(p)$, so the map $p \to \lambda^*(\theta, p)$ is continuous.

Remark 3.4. *In general, the uniqueness of optimizer is difficult to show for problem* (I.4). *When* $\alpha = 0$*, we provide below a uniqueness result for an approximated problem obtained by approximating the binomial distribution in* (I.4) *by the Poisson distribution* $Pois(\lambda p)$ *when* λ *is large enough and* p *is small :*

$$\lambda^*(\theta, p) \in \arg\max_{\lambda} \lambda\big(1 - Pois(\theta, \lambda, p)\big), \tag{I.9}$$

where $1 - Pois(\theta, \lambda, p) = \sum_{k=0}^{\theta-1} \frac{(\lambda p)^k e^{-\lambda p}}{k!}$.

Proposition 3.5 (Uniqueness)**.** *The approximated optimization problem* (I.9) *admits an unique solution* λ^*.

Proof.
Recall that $(\frac{(\lambda p)^k e^{-\lambda p}}{k!})' = p(\frac{(\lambda p)^{k-1} e^{-\lambda p}}{(k-1)!} - \frac{(\lambda p)^k e^{-\lambda p}}{k!})$. *Thus*

$$\begin{aligned}\frac{d[\lambda(1 - Pois(\theta, \lambda, p))]}{d\lambda} &= \sum_{k=0}^{\theta-1} \frac{(\lambda p)^k e^{-\lambda p}}{k!} - \lambda p e^{-\lambda p} + \lambda \sum_{k=1}^{\theta-1} p(\frac{(\lambda p)^{k-1} e^{-\lambda p}}{(k-1)!} - \frac{(\lambda p)^k e^{-\lambda p}}{k!}) \\ &= \sum_{k=0}^{\theta-1} \frac{(\lambda p)^k e^{-\lambda p}}{k!} - \lambda p \cdot \frac{(\lambda p)^{\theta-1} e^{-\lambda p}}{(\theta-1)!}\end{aligned}$$

The optimal λ^* *should satisfy*

$$\sum_{k=0}^{\theta-1} \frac{(\lambda p)^k}{k!} = \frac{(\lambda p)^\theta}{(\theta-1)!}.$$

We note that if $\lambda \geq \frac{\theta+1}{p}$*, then* $\frac{(\lambda p)^{k+1}}{(k+1)!} > \frac{(\lambda p)^k}{k!}$*, for* $k = 0, 1, \ldots \theta - 1$. *Thus,*

$$\sum_{k=0}^{\theta-1} \frac{(\lambda p)^k}{k!} < \theta \frac{(\lambda p)^\theta}{\theta!} = \frac{(\lambda p)^\theta}{(\theta-1)!}.$$

So we must have $\lambda^* < \frac{\theta+1}{p}$. *It follows that*

$$\frac{d^2[\lambda(1 - Pois(\theta, \lambda, p))]}{d\lambda^2} = -p \cdot \frac{(\lambda p)^{\theta-1} e^{-\lambda p}}{(\theta-1)!} - p(\theta) \frac{(\lambda p)^{\theta-1} e^{-\lambda p}}{(\theta-1)!} + p \frac{(\lambda p)^\theta e^{-\lambda p}}{(\theta-1)!} < 0$$

Thus $\lambda(1 - Pois(\theta, \lambda, p))$ *is concave in* λ *and uniqueness of the optimizer follows.*

Remark 3.6. Relation to the Nash equilibrium
We can relate the equilibrium of Definition 3.1 to a Nash equilibrium of the following game. Any player with threshold θ^0*, given the connectivity* $\{\lambda(\theta)\}_{\theta \neq \theta^0}$ *of the other players, computes its optimal connectivity*

$$\lambda^*(\theta^0, p) \in \arg\max_{\lambda} \lambda\big(1 - B^\alpha(\theta^0, \lambda, p)\big),$$

under the constraint that

$$\sum_{\theta\neq\theta^0}\mu(\theta)\frac{\lambda(\theta)}{\sum\limits_{\theta\neq\theta^0}\mu(\theta)\lambda(\theta)+\mu(\theta^0)\lambda^*(\theta^0)}B^\alpha(\theta,\lambda(\theta),p)+\mu(\theta^0)\frac{\lambda^*(\theta^0)}{\sum\limits_{\theta\neq\theta^0}\mu(\theta)\lambda(\theta)+\mu(\theta^0)\lambda^*(\theta^0)}B^\alpha(\theta^0,j,p)=p.$$

We can rewrite the above constraint $p = G(\lambda^*(\theta)_{\theta\neq\theta^0},\lambda)$ for some function G. Thus, the optimization criterion can be rewritten as :

$$\lambda^*(\theta)\in\arg\max_\lambda \lambda[1-B^\alpha(\lambda,\theta^0,G(\lambda^*(\theta)_{\theta\neq\theta^0},\lambda))].$$

In this sense, $(\lambda^*(\theta),\theta\geq 0)$ is a Nash equilibrium.

Theorem 3.1. *The equilibrium in the network of size n converges to the equilibrium in the limit case.*

Let $(T_n^*,(\lambda_n^*(\theta),\theta\geq 0))$ the standard equilibrium for network with size n

- *Given T_n^**
$$\lambda_n^*(\theta)\in\arg\max_\lambda \lambda\big(1-\frac{D_n^{\lambda,\theta}(T_n^*)}{n}\big),\quad \textit{for each } \theta; \tag{I.10}$$
- *T_n^* is the smallest solution of the fixed point equation :*
$$T_n^*=\sum_\theta \lambda_n^*(\theta)\mu_n(\theta)\cdot D_n^{\lambda_n^*,\theta}(T_n^*).$$

Suppose $\lambda_n^*(\theta)$ is unique in (I.10), if we define $p_n^* := \frac{T_n^*}{\sum_\theta \lambda_n^*(\theta)\mu_n(\theta)n}$, then we have

$$p_n^*\xrightarrow{n\to\infty}p^* \textit{ and } \lambda_n^*(\theta)\xrightarrow{n\to\infty}\lambda^*(\theta) \tag{I.11}$$

Proof. Firstly we notice that if $(T_n^*,(\lambda_n^*(\theta),\theta\geq 0))$ is the standard equilibrium of (I.10), then p_n^* is the fixed point of the map

$$\phi_n(p):=\sum_\theta\frac{\lambda_n^*(\theta,p)\mu_n(\theta)}{\sum_\theta\lambda_n^*(\theta,p)\mu_n(\theta)}\cdot\frac{D_n^{\lambda_n^*,\theta}(\lambda_n^*(\theta,p)\mu_n(\theta)n\cdot p)}{n}.\,\text{for}\, p\in[0,1] \tag{I.12}$$

By the continuity of arg max as proved in (3.3), we have $\lambda_n^*(\theta,p)\xrightarrow{n\to\infty}\lambda^*(\theta,p)$. Recall that uniform convergence results (I.29),we can obtain

$$\frac{D_n^{\lambda_n^*,\theta}(\lambda_n^*(\theta,p)\mu_n(\theta)n\cdot p)}{n}\xrightarrow{p}B^\alpha(\theta,\lambda,p).$$

These give $\phi_n(p)\xrightarrow{n\to\infty}\phi(p)$ in probability for $p\in[0,1]$. Obviously we have $|\phi_n(p)|\leq 1$ for each ϕ_n. If we furthermore assume the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is equicontinuous, i.e. for each $\epsilon>0$ there is a $\delta>0$ such that

$$|\phi_n(p_1)-\phi_n(p_2)|<\epsilon \tag{I.13}$$

whenever $|p_1-p_2|<\delta$ for all functions ϕ_n in the sequence. Then by Arzela-Ascoli theorem, the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ contains a uniformly convergent subsequence $\{\phi_{n_k}\}_{k\in\mathbb{N}}$. Now suppose $\{p_{n_k}\}_{k\in\mathbb{N}}$ are the fixed points of the maps $\{\phi_{n_k}\}_{k\in\mathbb{N}}$. i.e. $\phi_{n_k}(p_{n_k}) = 0$. Since $\{p_{n_k}\}_{k\in\mathbb{N}}$ are bounded in $[0,1]$, p^* is an accumulation point of p_{n_k}, that is, $p_{n_k}^{(i)} \to p^*$ over some subsequence. Taking limit of both side on subsequence, since ϕ_n is continuous in p, we obtain $\phi(p^*) = 0$. In particular, any accumulation point of the sequence $p_{n_k}^{(i)}$ is a zero point. Since the zero point is unique, the only accumulation point of the sequence p_{n_k} is the optimizer p^*. Thus we have proved that any subsequence of p_{n_k} has a further subsequence that converge to p^*, which indicates $p_{n_k} \to p^*$.

3.1 Analysis of the equilibrium

Case 1: The system is perfectly observable to all the nodes, i.e. all the θ and their distribution $\mu(\theta)$ are common knowledge. Then all the nodes will simultaneously choose their connectivity in the equilibrium state by rational expectations and the system simultaneously reaches the equilibrium $(\lambda^*(\theta), \theta \geq 0)$, by virtue of Proposition 3.2.

Case 2: The system is only partially observed by the players, for example each player only knows his own threshold θ. All the players optimally choose their linkage $\lambda^*(\theta, p^0)$ according to some anticipated link failure probability p^0 (this could be the failure probability estimated from the current environment). Then the link failure probability p^1 in the next stage is determined by the fixed point of the map $p \to F(p^0, p)$, where

$$F(p^0, p) := \sum_{\theta} \mu(\theta) \frac{\lambda^*(\theta, p^0)}{\sum_\theta \mu(\theta)\lambda^*(\theta, p^0)} B^\alpha(\theta, \lambda^*(\theta, p^0), p). \tag{I.14}$$

Then all nodes adjust their expectation of link failure probability from p^0 to p^1 and choose connectivity $\lambda^*(\theta, p^1)$ corresponding to p^1. This will be further updated to a series of link failure probabilities $\{p^0, p^1 ... p^n\}$.

We conjecture that the sequence $\{p^0, p^1, ..., p^n\}$ will not convergence to the equilibrium states p^*, since the following map H is divergent near the equilibrium state

$$H : p^0 \to p^1 = \min\{p \mid F(p^0, p) = p\}. \tag{I.15}$$

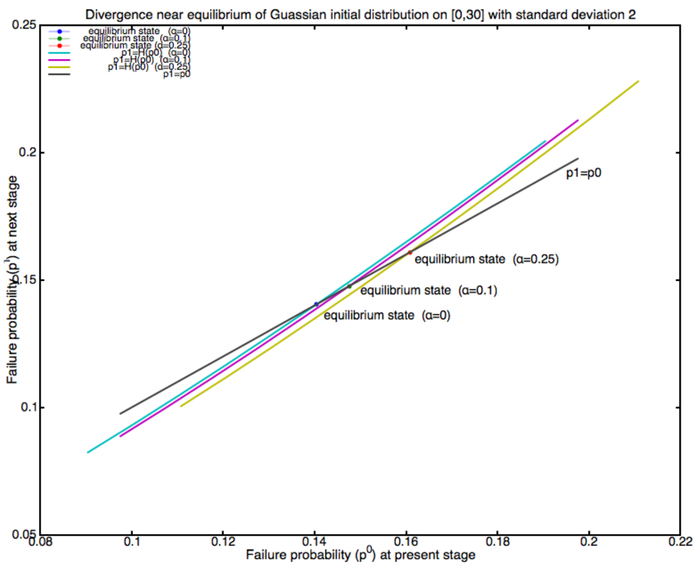
This means that the equilibrium is not stable in the sense that starting from the neighborhood of the equilibrium state, the above update of the rational expectations on the failure probability will not converge to the equilibrium.

Numerically we give examples of this instability phenomenon for different initial distributions and for different values of α:

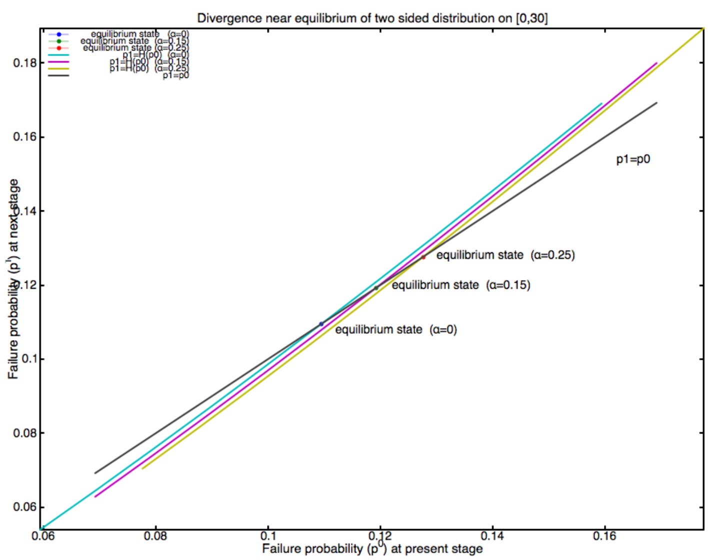
(i) Uniform initial distribution of θ on the interval $[0, 30]$ for $\alpha = 0, 0.15, 0.3$ respectively (Figure I.1c);

(ii) Gaussian initial distribution of θ on the interval $[0, 30]$ with mean 15 and standard deviation 2 for $\alpha = 0, 0.15, 0.2$ respectively (Figure I.1a);

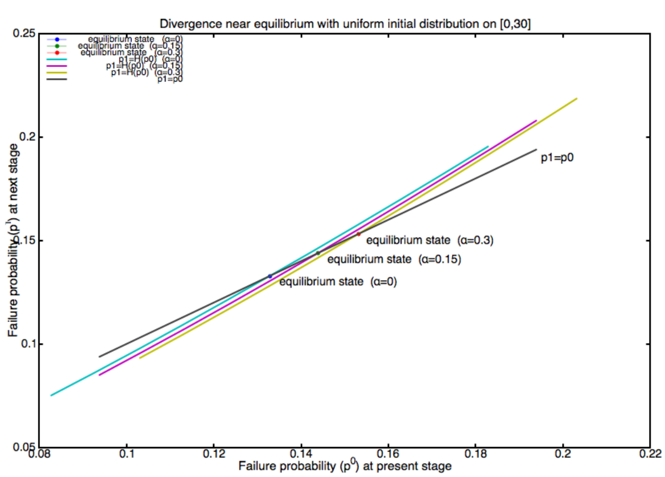
(iii) Two-sided distribution with $\mu(0) = 0.3$, $\mu(30) = 0.7$ and $\mu(\theta) = 0$ for $\theta \notin \{0, 30\}$ (Figure I.1b);



(a) Gaussian initial distribution of θ on $(0, 30)$ (with standard deviation 2)



(b) Two-sided distribution with $\mu(0) = 0.3$, $\mu(30) = 0.7$



(c) Uniform initial distribution of θ on $(0, 30)$

Figure I.1: Divergence near the equilibrium state for and for $\alpha = 0, 0.15, 0.3$. The map H has

4 Numerical results

In this section we discuss how the link failure probability in equilibrium changes with respect to the initial distribution and with respect to the growth rate α.

More precisely, we consider when different initial distributions on some range of the capacity θ where all distributions have the same mean but different variances. We choose θ in the interval $[0, 30]$ and we consider a Gaussian distribution on this range with mean 15 and standard deviation γ. We vary γ from 1 to 6, and in the case when $\alpha = 0$, we find that the link failure probability in equilibrium drops from 0.14072 to 0.13748. Similar situation happened for $\alpha > 0$ as well in the case when $\alpha = 0.15$ and $\alpha = 0.3$, where failure probability in equilibrium drops from 0.15187 to 0.14851 and from 0.16640 to 0.16295 respectively. This shows that relatively less concentration in the initial distribution leads to less failure probability in equilibrium, see Figure I.2. Moreover, the increase of the standard deviation in the initial distribution of θ leads to larger average connectivities in equilibrium for all three cases. This is consistent with the results in [1] that the average connectivity is a too simple statistics of network topology, see Figure I.3.

Remark 4.1. *When $\alpha = 0$, it is sufficient to consider failure probability in the equilibrium p^* as our measure of failure in the systems with respect to other variables. This is because another related measure : the final fraction of failure which is defined by*

$$F(p^*) := \sum_{\lambda,\theta} \mu(\lambda, \theta) \sum_{\theta} B(\theta, \lambda, p^*)$$

is an equivalent measure to p^*. This is because $F(p^*)$ is an increasing function p^* since $B(\theta, \lambda, p^*)$ is increasing with respect to p^*

Now we consider the case when $\alpha > 0$. We fix a Gaussian initial distribution for θ on the interval $[0, 30]$ with mean 15 and standard deviation 4.5461. We consider the growth rate varying from 0 to 0.35. We find the link failure probability in equilibrium and final fraction of failed nodes in equilibrium (see Theorem 1.2(ii)). Both increase from 0.13881 to 0.17021 and respectively from 0.14982 to 0.18299, see Figure I.4 and Figure I.5. This shows that higher growth/recovery rate in the system may result in larger failure probability and higher fraction of failed banks in the equilibrium.

This is intuitive: Note that when α increases, $\lambda^*(\theta, p), p)$ increases and $B^\alpha(\theta, \lambda^*(\theta, p), p)$ decreases. Thus the map $p \to \phi^\alpha(p)$ defined in (I.8) has the property that for $\alpha_1 \leq \alpha_2$, $\phi^{\alpha_1}(p) \geq \phi^{\alpha_2}(p)$, for each p. Thus the failure probability determined by relaxed fixed point of map $\phi^\alpha(p)$ tend to smaller.

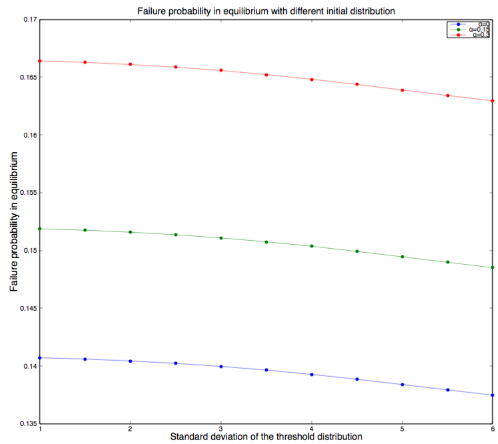


Figure I.2: Failure probability in equilibrium with different initial distribution of θ on $[0, 30]$. As the standard deviation parameter γ increases from 1 to 6, the failure probability in equilibrium drops from 0.14072 to 0.13748.

5 Conclusions

In this paper, we considered the optimal choice of connectivities by nodes in a large network. The nodes balance the benefits of connectivity with the risk of the contagion, which is endogenous and depends on the strategies of all nodes. We have studied the existence of an equilibrium in the system and the stability properties. Under full information, the equilibrium is reached simultaneously by all nodes, whereas under partial information, we show numerically that the equilibrium is unstable in the sense that a sequence of anticipated link failure probabilities does not converge. Our numerical results give the following insights

- A higher heterogeneity in the initial distribution of the threshold (as captured by its standard deviation) implies a lower default probability in equilibrium.
- A higher heterogeneity in the initial distribution of the threshold also gives a larger average connectivity in equilibrium.
- Systems with higher growth/recovery rates may lead to equilibria with higher failure probability as well as higher final fraction of failed banks.

The first point can be interpreted as a diversification effect at the system level: banks are dissimilar in terms of their thresholds. Combined, the first two points, demonstrate that the average connectivity is not by no means a predictor of the default probability. The last result is counter-intuitive. When the system has higher growth, the optimizing banks may engage in over-lending (in equilibrium) and this could lead to larger instability in the system.

6 Proofs and asymptotic results

In this section, we present the proofs of Theorem 1.2. In [1], the authors extend the differential equation method of Wormald (1995) and show that as the network size increases, the rescaled

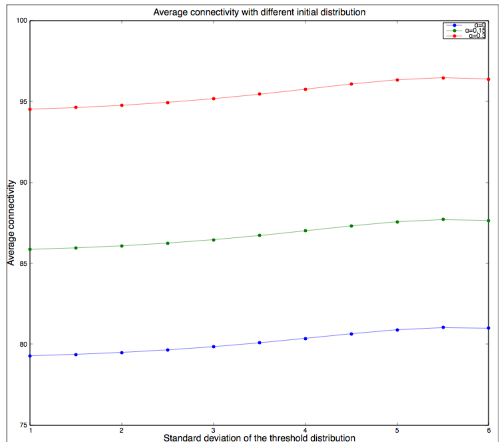


Figure I.3: The average connectivities of the system in the equilibrium with different initial distribution of θ on [0,30]. As the standard deviation parameter γ increases from 1 to 6, the average connectivity firstly increases from 79.295 to 81.029 then drops to 80.989.

Markov chains that describe the contagion converge in probability to a limit described by a system of ordinary differential equations. The case solved there corresponds to the zero growth case $\alpha = 0$ of this paper.

Here we show similar convergence results to a limit described by a more involved system of ordinary differential equations for the case $\alpha > 0$ and we obtain an analytical result on the final fraction of defaults in the network. The convergence result itself is shown in section 6.3 and relies on a careful partitioning of the Markov chain.

6.1 A Markov Chain Description of Contagion Dynamics

We describe now the contagion process on the financial system in terms of the dynamics of a lower dimension Markov chain. We partition the nodes according to their state of solvency, degree, threshold and number of defaulted neighbors. Let us define Let $S_n^{\theta,\lambda,l}(k)$ be the number of non-defaulted banks with initial threshold θ, λ outgoing links and l defaulted links at time T_k^n.
We introduce the additional variables of interest:

- $D_n^{\lambda,\theta}(k)$: the number of defaulted banks at time T_k^n with connectivity λ and initial threshold θ,

- $D_n(k)$: the number of defaulted banks at time T_k^n,

- $D_n^-(k)$: the number of unrevealed links belonging to defaulted banks at time T_k^n,

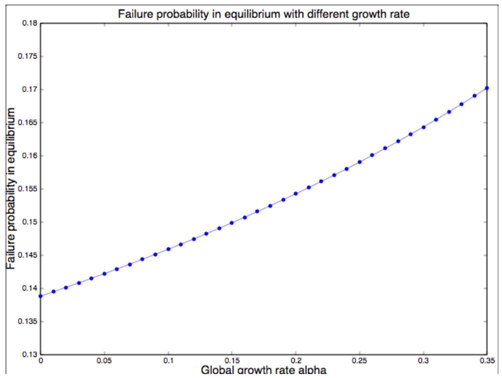


Figure I.4: Failure probability in equilibrium with different global growth rate with Gaussian initial distribution of θ. As the growth rate α increases from 0 to 0.35, the failure probability in equilibrium increases from 0.13881 to 0.17021.

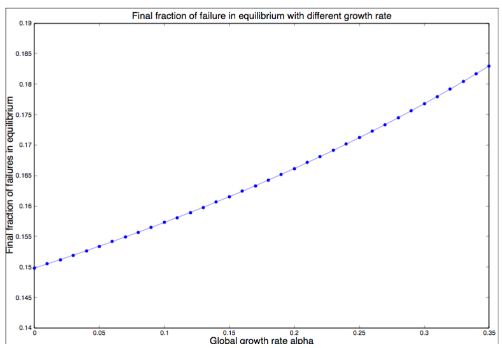


Figure I.5: Final fraction of failed banks in equilibrium with different global growth rate with Gaussian initial distribution of θ. As the growth rate α increases from 0 to 0.35, the final fraction of failure in equilibrium increases from 0.14982 to 0.18299.

for which it is easy to see that the following identities hold:

$$D_n^{\lambda,\theta}(k) \quad = \quad n\mu_n(\lambda,\theta) - \sum_{0\le l<[\theta+\frac{\alpha}{\lambda_n}\lambda T_k^n]} S_n^{\lambda,\theta,l}(k), \tag{I.16}$$

$$D_n^-(k) \quad = \quad \sum_{\lambda,0\le\theta\le\lambda} \lambda D_n^{\lambda,\theta}(k) - k, \tag{I.17}$$

$$D_n(k) \quad = \quad \sum_{\lambda,0\le\theta\le\lambda} D_n^{\lambda,\theta}(k). \tag{I.18}$$

The length of the default cascade is given by

$$T_n^{stop} = \inf\{0 \le k \le \bar{\lambda}_n n,\ D_n^-(k) = 0\} \wedge \bar{\lambda}_n n \tag{I.19}$$

where $\bar{\lambda}_n n$ represent the total number of links in the system.

The total number of defaults is given by $D_n(T_n^{stop})$, which represents the cardinal of the final set of defaulted nodes.

As shown in [1], by the random matching mechanism and coupling arguments, the dynamics of $\mathbf{S}_n(k) = \big(S_n^{\lambda,\theta,l}(k)\big)_{\lambda,0\le l\le\lambda,0\le\theta\le\lambda}$ represents a Markov chain with $(\mathcal{F}_k^n)_{k\ge0}$ being its natural filtration.

Let us now describe the transition probabilities of the Markov chain. For each step $1 \le k \le T_{stop}$, when a default link from the default node A is revealed, there are three possibilities for the partner B of an out-going edge of a defaulted node A at interaction time k:

(i) B is in default, the next state is $\mathbf{S}_n(k) = \mathbf{S}_n(k-1)$.

(ii) B is solvent, has degree λ and default threshold θ and this is the $(l+1)$-th deleted incoming edge and $l+1 < (\theta + \frac{\alpha\lambda T_k^n}{\bar\lambda_n}) \wedge \lambda$. The probability of this event is $\frac{(\lambda-l)S_n^{\lambda,\theta,l}(k)}{\bar\lambda_n n-k}$ and the changes for the next state will be

$$S_n^{\lambda,\theta,l}(k) = S_n^{\lambda,\theta,l}(k-1) - 1,$$
$$S_n^{\lambda,\theta,l+1}(k) = S_n^{\lambda,\theta,l+1}(k-1) + 1.$$

(iii) B is solvent, has degree λ and default threshold θ and this is the θ-th deleted out-going edge. Then with probability $\frac{(\lambda-\theta+1)S_n^{\lambda,\theta,\theta-1}(k)}{\bar\lambda_n n-k}$ we have

$$S_n^{\lambda,\theta,\theta-1}(k) = S_n^{\lambda,\theta,\theta-1}(k-1) - 1.$$

Let Δ_k be the difference operator: $\Delta_k S := S(k+1) - S(k)$. We obtain the following equations for the expectation of $\mathbf{S}_n(k+1)$, conditional on $\mathcal{F}_k^n$, by averaging over the possible transitions:

$$\mathbb{E}[\Delta_k S_n^{\lambda,\theta,0}|\mathcal{F}_k^n] \quad = \quad -\frac{\lambda S_n^{\lambda,\theta,0}(k)}{\bar\lambda_n n - k}$$

For $0 < l < \theta + \frac{\alpha}{\bar\lambda_n}\cdot\lambda\cdot T_k^n$,

$$\mathbb{E}[\Delta_k S_n^{\theta,\lambda,l}(k)|\mathcal{F}_k^n] = (\frac{\lambda-l+1}{\bar\lambda_n n-k})S_n^{\theta,\lambda,l-1}(k) - (\frac{\lambda-l}{\bar\lambda_n n-k})S_n^{\theta,\lambda,l}(k)$$

For $l = \theta + \frac{\alpha}{\bar\lambda_n}\cdot\lambda\cdot T_k^n$,

$$\mathbb{E}[\Delta_k S_n^{\theta,\lambda,l}(k)|\mathcal{F}_k^n] = (\frac{\lambda-l+1}{\bar\lambda_n n-k})S_n^{\theta,\lambda,l-1}(k) \tag{I.20}$$

For $l \ge \theta + \frac{\alpha}{\bar\lambda_n}\cdot\lambda\cdot T_k^n$,

$$S_n^{\theta,\lambda,l} = 0 \tag{I.21}$$

The initial condition is

$$S_n^{\lambda,\theta,l}(0) = \mu_n(\lambda,\theta)n\mathbb{1}(l=0)\mathbb{1}(0<\theta\le\lambda).$$

Equation (I.20) and (I.21) capture the feature that the default banks will not benefits from the threshold growth.

The above equations can be rewritten as following

$$\begin{cases} \mathbb{E}\big[\Delta_k S_n^{\theta,\lambda,l}(k)|\mathcal{F}_k^n\big] = (\frac{\lambda-l+1}{\lambda_n n-k})S_n^{\theta,\lambda,l-1}(k) - (\frac{\lambda-l}{\lambda_n n-k})S_n^{\theta,\lambda,l}(k) & \text{for } k \geq k_{min}^{\theta,l,n} \\ S_n^{\theta,\lambda,l}(k) = 0 & \text{for } k < k_{min}^{\theta,l,n} \\ S_n^{\theta,\lambda,l}(k_{min}^{\theta,l,n}) = S_n^{\theta,\lambda,l}(0) & \text{for } k_{min}^{\theta,l,n} = 0, l = 0 \\ S_n^{\theta,\lambda,l}(k_{min}^{\theta,l,n}) = 0 & \text{for } k_{min}^{\theta,l,n} = 0, l \neq 0 \end{cases} \tag{SDE}$$

where $k_{min}^{\theta,l,n} = \inf\{k | l \leq \lambda \wedge (\theta + \frac{\alpha\lambda}{\lambda_n} \cdot T_k^n)\}$. The interpretation of k_{min} is the first interaction time when a bank starting with threshold θ has accumulated enough cash flows to withstand $l \leq \lambda$ withdrawn credit lines.
We also remark that for $l \leq \theta$, the bank can withstand the withdrawn credit lines using only the initial threshold, so we have $k_{min} = 0$.

6.2 A Law of Large Numbers for the Contagion Process

We now show in this section that the path of $\mathbf{S}_n(k)$ for $k \leq T_n^{stop}$ is, with high probability, close to the solution of the ordinary differential equations stated below.

We define the following set of differential equations:

$$\begin{cases} \frac{ds^{\theta,\lambda,l}}{dt}(t) = (\frac{\lambda-l+1}{\lambda-u})s^{\theta,\lambda,l-1}(t) - (\frac{\lambda-l}{\lambda-t})s^{\theta,\lambda,l}(t) & \text{for } t \geq t_{min}^{\theta,l} \\ s^{\theta,\lambda,l}(t) = 0 & \text{for } t < t_{min}^{\theta,l} \\ s^{\theta,\lambda,l}(t_{min}^{\theta,l}) = s^{\theta,\lambda,l}(0) & \text{for } t_{min}^{\theta,l} = 0, l = 0 \\ s^{\theta,\lambda,l}(t_{min}^{\theta,l}) = 0 & \text{for } t_{min}^{\theta,l} = 0, l \neq 0 \end{cases} \tag{DE}$$

With initial conditions

$$s^{\lambda,\theta,l}(0) = \mu(\lambda,\theta)\mathbb{1}(l=0)\mathbb{1}(0 < \theta \leq \lambda).$$

where

$$t_{min}^{\theta,l} = \inf\{t | l \leq \lambda \wedge (\theta + \frac{\alpha\lambda}{\bar{\lambda}} t)\}$$

and $\bar{\lambda} := \sum_{\theta,\lambda} \lambda\mu(\theta,\lambda)$. In the limit case, $t_{min}^{\theta,l}$ is characterised by

$$\theta + t_{min}\frac{\alpha\lambda}{\bar{\lambda}} = l.$$

This gives $t_{min}^{\theta,l} = \frac{(l-\theta)\bar{\lambda}}{\alpha\lambda}$, for $\theta \leq l \leq \lambda$ and for $l \leq \theta$, $t_{min} = 0$.

Lemma 6.1. *The system of ordinary differential equations (DE) admits the unique solution*

$$\mathbf{s}(t) = \big(s^{\lambda,\theta,l}(t)\big)_{\lambda, 0 \leq l \leq \lambda, 0 \leq \theta \leq \lambda}$$

in the interval $0 \leq t < \bar{\lambda}$,*with*

(i) $l \leq \theta$,

$$s^{\theta,\lambda,l}(t) = \binom{\lambda}{l}(1-\frac{t}{\bar{\lambda}})^{\lambda-l}(\frac{t}{\bar{\lambda}})^{l}\mu(\lambda,\theta). \tag{I.22}$$

(ii) $l = \theta + k$ *with* $k \geq 1$,
for $0 \leq t \leq t_{min}^{\theta,\theta+k}$

$$s^{\theta,\lambda,l}(t) = 0. \tag{I.23}$$

for $t > t_{min}^{\theta,\theta+k}$

$$s^{\theta,\lambda,l}(t) = \Bigg[\sum_{r\leq\theta}\binom{\lambda-r}{l-r}(1-\frac{t-t_{min}^{\theta,\theta+k}}{\bar{\lambda}})^{\lambda-l}(\frac{t-t_{min}^{\theta,\theta+k}}{\bar{\lambda}})^{l-r}\binom{\lambda}{r}(1-\frac{t_{min}^{\theta,\theta+k}}{\bar{\lambda}})^{\lambda-r}(\frac{t_{min}^{\theta,\theta+k}}{\bar{\lambda}})^{r}$$
$$+\sum_{1\leq m\leq k-1}\ \sum_{1\leq u^1\leq\ldots u^m\leq k-1}\ \sum_{r\leq\theta}\binom{\lambda-\theta-u^m}{l-\theta-u^m}(1-\frac{t-t_{min}^{\theta,\theta+k}}{\bar{\lambda}})^{\lambda-l}(\frac{t-t_{min}^{\theta,\theta+k}}{\bar{\lambda}})^{l-\theta-u^m} \tag{I.24}$$

$$\binom{\lambda-\theta-u^{m-1}}{\theta+u^m-\theta-u^{m-1}}(1-\frac{t_{min}^{\theta,\theta+k}-t_{min}^{\theta,\theta+u^m}}{\bar{\lambda}})^{\lambda-\theta-u^m}(\frac{t_{min}^{\theta,\theta+k}-t_{min}^{\theta,\theta+u^m}}{\bar{\lambda}})^{\theta+u^m-\theta-u^{m-1}}\ldots$$
$$\binom{\lambda-u^1-\theta}{\theta+u^2-\theta-u^1}(1-\frac{t_{min}^{\theta,\theta+u^3}-t_{min}^{\theta,\theta+u^2}}{\bar{\lambda}})^{\lambda-\theta-u^2}(\frac{t_{min}^{\theta,\theta+u^3}-t_{min}^{\theta,\theta+u^2}}{\bar{\lambda}})^{\theta+u^2-\theta-u^1}$$
$$\binom{\lambda-r}{\theta+u^1-r}(1-\frac{t_{min}^{\theta,\theta+u^2}-t_{min}^{\theta,\theta+u^1}}{\bar{\lambda}})^{\lambda-\theta-u^1}(\frac{t_{min}^{\theta,\theta+u^2}-t_{min}^{\theta,\theta+u^1}}{\bar{\lambda}})^{\theta+u^1-r}$$
$$\binom{\lambda}{r}(1-\frac{t_{min}^{\theta,\theta+u^1}}{\bar{\lambda}})^{\lambda-r}(\frac{t_{min}^{\theta,\theta+u^1}}{\bar{\lambda}})^{r}\Bigg]\mu(\lambda,\theta). \tag{I.25}$$

In our case when $t_{min}^{\theta,l} = \frac{(l-\theta)\bar{\lambda}}{\alpha\lambda}$, *for* $\theta \leq l \leq \lambda$, *we have*

$$s^{\theta,\lambda,l}(t) = \Bigg[\sum_{r\leq\theta}\binom{\lambda-r}{l-r}(1-\frac{t}{\bar{\lambda}}+\frac{k}{\alpha\lambda})^{\lambda-l}(\frac{t}{\bar{\lambda}}-\frac{k}{\alpha\lambda})^{l-r}\binom{\lambda}{r}(1-\frac{k}{\alpha\lambda})^{\lambda-r}(\frac{k}{\alpha\lambda})^{r}$$
$$+\sum_{1\leq m\leq k-1}\ \sum_{1\leq u^1\leq\ldots u^m\leq k-1}\ \sum_{r\leq\theta}\binom{\lambda-\theta-u^m}{l-\theta-u^m}(1-\frac{t}{\bar{\lambda}}+\frac{k}{\alpha\lambda})^{\lambda-l}(\frac{t}{\bar{\lambda}}-\frac{k}{\alpha\lambda})^{l-\theta-u^m}$$
$$\binom{\lambda-\theta-u^{m-1}}{\theta+u^m-\theta-u^{m-1}}(1-\frac{k-u^m}{\alpha\lambda})^{\lambda-\theta-u^m}(\frac{k-u^m}{\alpha\lambda})^{\theta+u^m-\theta-u^{m-1}}\ldots$$
$$\binom{\lambda-u^1-\theta}{\theta+u^2-\theta-u^1}(1-\frac{u^3-u^2}{\alpha\lambda})^{\lambda-\theta-u^2}(\frac{u^3-u^2}{\alpha\lambda})^{\theta+u^2-\theta-u^1}$$
$$\binom{\lambda-r}{\theta+u^1-r}(1-\frac{u^2-u^1}{\alpha\lambda})^{\lambda-\theta-u^1}(\frac{u^2-u^1}{\alpha\lambda})^{\theta+u^1-r}\binom{\lambda}{r}(1-\frac{u^1}{\alpha\lambda})^{\lambda-r}(\frac{u^1}{\alpha\lambda})^{r}\Bigg]\mu(\lambda,\theta). \tag{I.26}$$

Proof.

(i) For the case $l \leq \theta$, the proof is given by Lemma 5.8 in [1].

(ii) For the case $l \geq \theta + k$. Firstly by the definition of equation (DE), we obtain (I.23). Now we prove (I.24) the results by induction. When $l = \theta + 1$, For $t > t_{min}^{\theta,\theta+1}$, by setting $t_{min}^{\theta,\theta+1}$ as the initial time and applying Lemma 6.3, we have

$$\begin{aligned} s^{\theta,\lambda,l}(t) &= \sum_{r \leq \theta} \binom{\lambda - r}{l - r} (1 - \frac{t - t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{\lambda - l} (\frac{t - t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{l-r} s^{\theta,\lambda,r}(\frac{t_{min}^{\theta,\theta+1}}{\bar{\lambda}}) \\ &= \sum_{r \leq \theta} \binom{\lambda - r}{l - r} (1 - \frac{t - t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{\lambda - l} (\frac{t - t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{l-r} \binom{\lambda}{r} (1 - \frac{t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{\lambda - r} (\frac{t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^r s_0^{\theta,\lambda,0}. \end{aligned}$$

where the second equality is due to the results in (i) for $(s^{\theta,\lambda,r})_{r \leq \theta}$.

(iii) When $l = \theta + 2$, we have similarly by Lemma 6.3,

$$\begin{aligned} s^{\theta,\lambda,l}(t) &= \sum_{r \leq \theta+1} \binom{\lambda - r}{l - r} (1 - \frac{t - t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{\lambda - l} (\frac{t - t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{l-r} s^{\theta,\lambda,r}(\frac{t_{min}^{\theta,\theta+2}}{\bar{\lambda}}) \\ &= \sum_{r \leq \theta} \binom{\lambda - r}{l - r} (1 - \frac{t - t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{\lambda - l} (\frac{t - t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{l-r} \binom{\lambda}{r} (1 - \frac{t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{\lambda - r} (\frac{t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^r s_0^{\theta,\lambda,0} \\ &+ \binom{\lambda - \theta - 1}{l - \theta - 1} (1 - \frac{t - t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{\lambda - l} (\frac{t - t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{l-\theta-1} s^{\theta,\lambda,\theta+1}(\frac{t_{min}^{\theta,\theta+2}}{\bar{\lambda}}). \end{aligned}$$

Since $t_{min}^{\theta,\theta+2} > t_{min}^{\theta,\theta+1}$, now by the results in (ii) for $s^{\theta,\lambda,\theta+1}$.

$$\begin{aligned} s^{\theta,\lambda,\theta+1}(\frac{t_{min}^{\theta,\theta+2}}{\bar{\lambda}}) = \sum_{r \leq \theta} & \binom{\lambda - r}{\theta + 1 - r} (1 - \frac{t_{min}^{\theta,\theta+2} - t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{\lambda - \theta - 1} (\frac{t_{min}^{\theta,\theta+2} - t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{\theta+1-r} \\ & \binom{\lambda}{r} (1 - \frac{t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{\lambda - r} (\frac{t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^r s_0^{\theta,\lambda,0}. \end{aligned}$$

which gives for $t > t_{min}^{\theta,\theta+2}$,

$$\begin{aligned} s^{\theta,\lambda,l}(t) &= \sum_{r \leq \theta} \binom{\lambda - r}{l - r} (1 - \frac{t - t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{\lambda - l} (\frac{t - t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{l-r} \binom{\lambda}{r} (1 - \frac{t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{\lambda - r} (\frac{t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^r s_0^{\theta,\lambda,0} \\ &+ \binom{\lambda - \theta - 1}{l - \theta - 1} (1 - \frac{t - t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{\lambda - l} (\frac{t - t_{min}^{\theta,\theta+2}}{\bar{\lambda}})^{l-\theta-1} \\ & \sum_{r \leq \theta} \binom{\lambda - r}{\theta + 1 - r} (1 - \frac{t_{min}^{\theta,\theta+2} - t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{\lambda - \theta - 1} (\frac{t_{min}^{\theta,\theta+2} - t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{\theta+1-r} \\ & \binom{\lambda}{r} (1 - \frac{t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^{\lambda - r} (\frac{t_{min}^{\theta,\theta+1}}{\bar{\lambda}})^r s_0^{\theta,\lambda,0}. \end{aligned}$$

(iv) Suppose up to $l=\theta+k$, (I.24) is true. Now for $l=\theta+k+1$,

$$s^{\theta,\lambda,l}(t) = \sum_{r\leq\theta+k}\binom{\lambda-r}{l-r}\left(1-\frac{t-t_{min}^{\theta,\theta+k+1}}{\bar{\lambda}}\right)^{\lambda-l}\left(\frac{t-t_{min}^{\theta,\theta+k+1}}{\bar{\lambda}}\right)^{l-r}s^{\theta,\lambda,r}\left(\frac{t_{min}^{\theta,\theta+k+1}}{\bar{\lambda}}\right) \tag{I.27}$$

$$=\sum_{r\leq\theta}\binom{\lambda-r}{l-r}(1-\frac{t-t_{min}^{\theta,\theta+k+1}}{\bar{\lambda}})^{\lambda-l}(\frac{t-t_{min}^{\theta,\theta+k+1}}{\bar{\lambda}})^{l-r}s^{\theta,\lambda,r}(\frac{t_{min}^{\theta,\theta+k+1}}{\bar{\lambda}})$$

$$+\sum_{\theta+1\leq r\leq\theta+k}\binom{\lambda-r}{l-r}(1-\frac{t-t_{min}^{\theta,\theta+k+1}}{\bar{\lambda}})^{\lambda-l}(\frac{t-t_{min}^{\theta,\theta+k+1}}{\bar{\lambda}})^{l-r}s^{\theta,\lambda,r}(\frac{t_{min}^{\theta,\theta+k+1}}{\bar{\lambda}}).$$

After plugging the formula of $\left(s^{\theta,\lambda,r}(\frac{t_{min}^{\theta,\theta+k+1}}{\bar{\lambda}})\right)_{r\leq\theta+k}$ into (I.27), we obtain (I.24) also holds for $l=\theta+k+1$. Thus by mathematical induction.

Remark 6.2. *Let us now compare the solution to* (DE) *to the solution of another system of ODEs that describe a modified system where we allow failed banks to receive grows from their links and recover. The latter system is given by*

$$\begin{cases}\frac{ds^{\theta,\lambda,l}}{dt}(t)=(\frac{\lambda-l+1}{\bar{\lambda}-u})s^{\theta,\lambda,l-1}(t)-(\frac{\lambda-l}{\bar{\lambda}-t})s^{\theta,\lambda,l}(t)\\ s^{\lambda,\theta,l}(0)=\mu(\lambda,\theta)\mathbb{1}(l=0)\mathbb{1}(0<\theta\leq\lambda)\end{cases} \tag{I.28}$$

and its solution is standard $\binom{\lambda}{l}(1-\frac{t}{\bar{\lambda}})^{\lambda-l}(\frac{t}{\bar{\lambda}})^{l}\mu(\lambda,\theta)$ *(as given in Lemma 6.3 below).*

We check

$$s^{\lambda,\theta,l}(t)\leq\binom{\lambda}{l}(1-\frac{t}{\bar{\lambda}})^{\lambda-l}(\frac{t}{\bar{\lambda}})^{l}\mu(\lambda,\theta).$$

Lemma 6.3. *[1]The system of ordinary differential equations admits the unique solution*

$$\frac{ds^{\theta,\lambda,l}}{dt}(t)=(\frac{\lambda-l+1}{\bar{\lambda}-u})s^{\theta,\lambda,l-1}(t)-(\frac{\lambda-l}{\bar{\lambda}-t})s^{\theta,\lambda,l}(t),\ \textit{for } t\geq t_0$$

admits the unique solution

$$\mathbf{s}(t)=\left(s^{\lambda,\theta,l}(t)\right)_{\lambda,0\leq l\leq\lambda,0\leq\theta\leq\lambda}$$

in the interval $t_0\leq t<\bar{\lambda}$*, which satisfies*

$$s^{\theta,\lambda,l}(t)=\binom{\lambda}{l}(1-\frac{t-t_0}{\bar{\lambda}})^{\lambda-l}(\frac{t-t_0}{\bar{\lambda}})^{l}s^{\theta,\lambda,l}(t_0)$$

6.3 Proof of Theorem 1.2

Proof. We now proceed to the proof of Theorem 1.2 whose aim is to approximate the value $\frac{D_n(T_n^{stop})}{n}$ as $n\to\infty$. We build on the techniques used in Lemma 6.4 and Theorem 3.8 in [1]. In contrast to [1], we need a careful partitioning of the Markov chain. We first prove the convergence for the rescaled number of banks that have $l<\theta$ i.e. which are guaranteed to survive. This part of the proof follows Theorem 3.8 in [1]. Next we consider $l\geq\theta$ and partition the time interval according to the possibility that growth is sufficient for survival. For every

$l \geq \theta$ there is a minimal time $t_{min}^{\theta,l}$ such that a bank with initial threshold θ can survive after $t_{min}^{\theta,l}$. As the induction initial step convergence on the entire time interval $[0,1]$ of the rescaled vector $S_n^{\lambda,\theta,l}$ for $l < \theta$. As induction step, we show that convergence on the interval $[t_{min}^{\theta,\hat{l}}, 1]$ of $S_n^{\lambda,\theta,l}$ for $l < \hat{l}$ implies convergence on the interval $[t_{min}^{\theta,\hat{l}+1}, 1]$ of $S_n^{\lambda,\theta,l}$ for $l < \hat{l}+1$.

To simplify the analysis we assume without loss of generality that there is an upper bound on λ, i.e, $\exists K$, such that $\mu(\lambda,\theta) = 0$ for all $\lambda \geq K$. For $K \geq 1$, we denote

$$\mathbf{s}^K := \left(s^{\lambda,\theta,l}(t)\right)_{\lambda \leq K,\ 0 \leq l,\theta \leq \lambda} \text{ and}$$
$$\mathbf{S}_n^K := \left(S_n^{\lambda,\theta,l}(k)\right)_{\lambda \leq K,\ 0 \leq l,\theta \leq \lambda}.$$

We now show by induction that

$$\sup_{0 \leq t \leq \hat{\tau} n} \left|\mathbf{S}_n^K(k)/n - \mathbf{s}^K(k/n)\right| \leq C\epsilon + o_p(1) \tag{I.29}$$

As the induction first step, we consider the subvector with $l < \theta$.

For $K \geq 1$, we denote this subvector where $l < \theta$ by

$$\mathbf{s}^{K,\theta} := \left(s^{\lambda,\theta,l}(t)\right)_{\lambda \leq K,\ 0 \leq l < \theta \leq \lambda} \text{ and}$$
$$\mathbf{S}_n^{K,\theta} := \left(S_n^{\lambda,\theta,l}(k)\right)_{\lambda \leq K,\ 0 \leq l < \theta \leq \lambda},$$

where the superscript θ marks the upper bound on l. For an arbitrary constant $\epsilon > 0$, we define the domain U_ϵ^θ as

$$U_\epsilon^\theta = \{\left(\tau, s^K\right) \in \mathbb{R}^{K\theta+1} \ : \ -\epsilon < \tau < \bar{\lambda} - \epsilon\ ,\ -\epsilon < s^{\lambda,k,\theta,l} < 1\}.$$

We now verify the conditions of Lemma 6.4. The domain U_ϵ^θ is a bounded open set which contains the support of all initial values of the variables. Each variable is bounded by a constant times n ($C_0 = 1$). By the definition of our process, the Boundedness condition is satisfied with $\beta(n) = 1$. The second condition of the theorem is satisfied by some $\gamma_1(n) = O(1/n)$. Finally the Lipschitz property is also satisfied since $\bar{\lambda} - \tau$ is bounded away from zero. Then by Lemma 6.4, for a sufficiently large constant C, we have

$$\mathbb{P}(\forall k \leq n\sigma_C(n), \mathbf{S}_n^{K,\theta}(k) = n\mathbf{s}^{K,\theta}(k/n) + O(n^{3/4})) = 1 - O(b(K)n^{-1/4}exp(-n^{-1/4}))$$

uniformly for all $0 \leq k \leq n\sigma_C(n)$ where

$$\sigma_C(n) = \sup\{t \geq 0, d(\mathbf{s}^{K,\theta}(t), \partial U_\epsilon\ ) \geq Cn^{-1/4}\}.$$

which means that

$$\sup_{0 \leq k \leq \hat{\tau} n} \left|\mathbf{S}_n^{K,\theta}(k)/n - \mathbf{s}^{K,\theta}(k/n)\right| \leq C\epsilon + o_p(1)$$

with $\hat{\tau} = \bar{\lambda} - \epsilon$.

For $\hat{l} \geq \theta$ we denote the subvector $(s^{K,\hat{l}}, S_n^{K,\hat{l}})_{\lambda \leq K,\ 0 \leq l < \hat{l}}$ with index $l < \hat{l}$ by

$$\mathbf{s}^{K,\hat{l}} := \left(s^{\lambda,\theta,l}(t)\right)_{\lambda \leq K,\ 0 \leq l < \hat{l}} \text{ and}$$
$$\mathbf{S}_n^{K,\hat{l}} := \left(S_n^{\lambda,\theta,l}(k)\right)_{\lambda \leq K,\ 0 \leq l < \hat{l}},$$

Now consider on the domain

$$U_\epsilon^{\hat{l}} = \{\left(t, s^K\right) \in \mathbb{R}^{K\hat{l}+1} \ : \ -\epsilon < t < \bar{\lambda} - \epsilon \ , \ -\epsilon < s^{\lambda,k,\theta,l} < 1\}.$$

And we prove by induction on $\hat{l}$ that

$$\sup_{0 \leq k \leq \hat{\tau} n} \left| \mathbf{S}_n^{K,\hat{l}}(k)/n - \mathbf{s}^{K,\hat{l}}(k/n) \right| \leq C^{\hat{l}} \epsilon + o_p(1) \tag{I.30}$$

We have seen that the above holds for $\hat{l} \leq \theta - 1$, so we use this as the induction initial step. We now suppose that it holds for an $\hat{l} \geq \theta - 1$ and we need to show that it holds for $\hat{l} + 1$. We consider the ODEs solution $\tilde{\mathbf{s}}^{K,\theta} := \left(\tilde{s}^{\lambda,\theta,l}(t)\right)_{\lambda < K,\ 0 \leq l < \hat{l}+1}$ on the domain

$$U_\epsilon^{\hat{l}+1} = \{\left(t, s^K\right) \in \mathbb{R}^{K(\hat{l}+1)+1} \ : \ -\epsilon \leq t < \bar{\lambda} - \epsilon \ , \ -\epsilon < s^{\lambda,k,\theta,l} < 1\}.$$

with initial condition $\tilde{\mathbf{s}}_n^{K,\hat{l}+1}(k_{min}^{\theta,\hat{l}+1,n}/n) = (\mathbf{S}_n^{K,\hat{l}}(k_{min}^{\theta,\hat{l}+1,n})/n, 0)$, namely we start the same system from the time $k_{min}^{\theta,\hat{l}+1,n}/n$. At this time it is guaranteed that $S_n^{K,\hat{l}+1} = 0$.

Then from Lemma 6.4, we have

$$\sup_{k_{min}^{\theta,\hat{l}+1,n} \leq k \leq \hat{\tau} n} \left| \mathbf{S}_n^{K,\hat{l}+1}(k)/n - \tilde{\mathbf{s}}^{K,\hat{l}+1}(k/n) \right| \leq C\epsilon + o_p(1) \tag{I.31}$$

By the induction hypothesis, namely (I.30), we have $\left| \mathbf{S}_n^{K,\hat{l}}(k_{min}^{\theta,\hat{l}+1,n})/n - \mathbf{s}^{K,\hat{l}}(k_{min}^{\theta,\hat{l}+1,n}/n) \right| \leq$ $C^{\hat{l}}\epsilon + o_p(1)$
and using that gives

$$\left| \tilde{\mathbf{s}}^{K,\hat{l}}(k_{min}^{\theta,\hat{l}+1,n}/n) - \mathbf{s}^{K,\hat{l}}(k_{min}^{\theta,\hat{l}+1,n}/n) \right| \leq C^{\hat{l}}\epsilon + o_p(1). \tag{I.32}$$

By definition

$$s^{\theta,\lambda,\hat{l}+1}(t) = 0 \text{ for } t \leq t_{min}^{\theta,\hat{l}+1}$$

Now since

$$\mathbb{E}(T_k^n) = \sum_{i=1}^k \mathbb{E}(T_i^n - T_{i-1}^n) = \sum_{i=1}^k \frac{1}{n} = \frac{k}{n}$$

$$Var(T_k^n) = \sum_{i=1}^k Var(T_i^n - T_{i-1}^n) = \sum_{i=1}^k \frac{1}{n^2} = \frac{k}{n^2} \sim o(\frac{1}{n})$$

we have by Chebysev's inequality, as $n \to \infty$, in probability

$$T_k^n \xrightarrow{p} \frac{k}{n}$$

This gives that $k_{min}^{\theta,\hat{l}+1}/n = t_{min}^{\theta,\hat{l}+1} + o_p(1)$. Thus by continuity property of ODEs,

$$\left|\tilde{s}^{\theta,\lambda,\hat{l}+1}(k_{min}^{\theta,\hat{l}+1,n}/n) - s^{\theta,\lambda,\hat{l}+1}(k_{min}^{\theta,\hat{l}+1,n}/n)\right| \leq C\epsilon + o_p(1)$$

. Combining with equation (II.16) we obtain

$$\left|\tilde{\mathbf{s}}^{K,\hat{l}+1}(k_{min}^{\theta,\hat{l}+1,n}/n) - \mathbf{s}^{K,\hat{l}+1}(k_{min}^{\theta,\hat{l}+1,n}/n)\right| \leq C_1\epsilon + o_p(1).$$

Thus by the stability results of ODEs we have

$$\sup_{k_{min}^{\theta,\hat{l}+1,n} \leq k \leq \hat{\tau}n} \left|\mathbf{s}_n^{K,\hat{l}+1}(k/n) - \tilde{\mathbf{s}}^{K,\hat{l}+1}(k/n)\right| \leq C_2\epsilon + o_p(1)$$

Combined with (I.31) this gives

$$\sup_{k_{min}^{\theta,\hat{l}+1,n} \leq k \leq \hat{\tau}n} \left|\mathbf{S}_n^{K,\hat{l}+1}(k)/n - \mathbf{s}^{K,\hat{l}+1}(k/n)\right| \leq (C_2 + C)\epsilon + o_p(1) \tag{I.33}$$

By definition

$$S_n^{\theta,\lambda,\hat{l}+1}(k) = 0 \text{ for } k \leq k_{min}^{\theta,\hat{l}+1,n}$$

and

$$s^{\theta,\lambda,\hat{l}+1}(t) = 0 \text{ for } t \leq t_{min}^{\theta,\hat{l}+1}$$

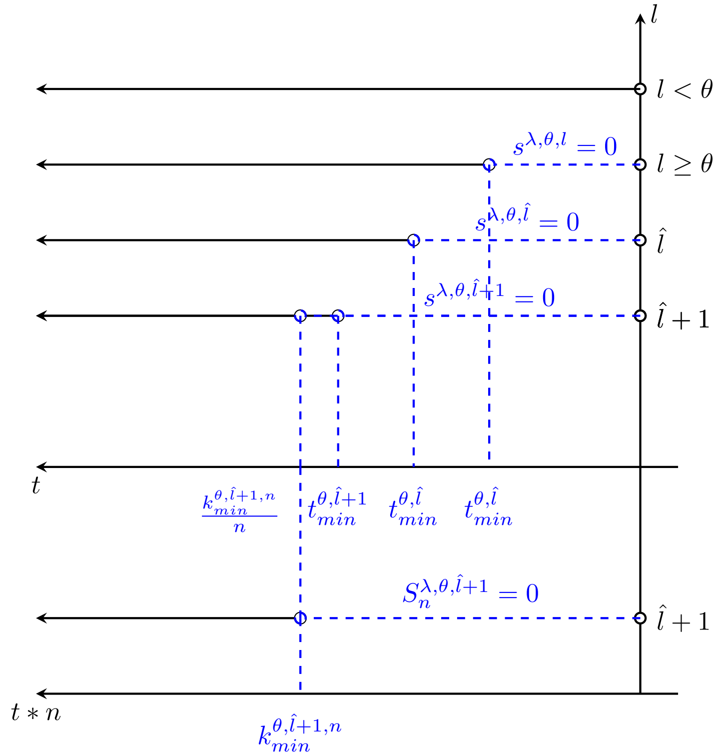
Combining with (I.30), it gives

$$\sup_{0 \leq k \leq \min\{t_{min}^{\theta,\hat{l}+1}n, k_{min}^{\theta,\hat{l}+1,n}\}} \left|\mathbf{S}_n^{K,\hat{l}+1}(k)/n - \mathbf{s}^{K,\hat{l}+1}(k/n)\right| = 0. \tag{I.34}$$

Thus by the continuity of solution $\left(s^{\theta,\lambda,\hat{l}+1}(t)\right)_{t\geq 0}$ between $\min\{t_{min}^{\theta,\hat{l}+1}n, k_{min}^{\theta,\hat{l}+1,n}\}$ and $k_{min}^{\theta,\hat{l}+1,n}$, combining with (I.30), (I.33) and (I.34), we obtain

$$\sup_{0 \leq k \leq \hat{\tau}n} \left|\mathbf{S}_n^{K,\hat{l}+1}(k)/n - \mathbf{s}^{K,\hat{l}+1}(k/n)\right| \leq (C_2 + C)\epsilon + o_p(1)$$

Thus by mathematical induction, we have (I.29) hold.



We now determine the stopping time of the cascade, the fraction of failed banks and conclude the proof of the theorem. Let the following variables denote the rescaled number of failed banks and respectively failed links (in the limit)

$$\begin{aligned}
d^{\lambda,\theta}(t) &= \mu(\lambda,\theta) - \sum_{0 \leq l < [\theta + \frac{\alpha\lambda t}{\lambda}]} s^{\lambda,\theta,l}(t), \\
d^{-}(t) &= \sum_{\lambda, 0 \leq \theta \leq \lambda} \lambda d^{\lambda,\theta}(t) - t, \\
d(t) &= \sum_{\lambda, 0 \leq \theta \leq \lambda} d^{\lambda,\theta}(t).
\end{aligned}$$

We notice that

$$\begin{aligned}
\sum_{\lambda, 0 \leq \theta \leq \lambda} d^{\lambda,\theta}(t) &= \sum_{\lambda, 0 \leq \theta \leq \lambda} \left[\mu(\lambda,\theta) - \sum_{0 \leq l < \left[\theta + \frac{\alpha\lambda t}{\lambda}\right]} s^{\lambda,\theta,l}(t) \right] \\
&= \sum_{\lambda, 0 \leq \theta \leq \lambda} \mu(\lambda,\theta) \left[1 - \sum_{0 \leq l < [\theta + \frac{\alpha\lambda t}{\lambda}]} \beta^{\lambda,\theta,l}(\frac{t}{\bar{\lambda}}) \right]
\end{aligned}$$

Thus we have

$$d^{-}(t)=\bar{\lambda}(\sum_{\lambda,0\leq\theta\leq\lambda}\frac{\lambda}{\bar{\lambda}}d^{\lambda,\theta}(t)-\frac{t}{\bar{\lambda}})=\bar{\lambda}(J(\frac{t}{\bar{\lambda}})-\frac{t}{\bar{\lambda}})$$

The length of the default cascade is given by

$$t^{stop}=\inf\{0\leq t\leq\bar{\lambda},\ d^{-}(t)\leq 0\}\wedge\bar{\lambda}$$

By definition we have

$$\begin{aligned}\left|D_n^{-}(k)/n-d^{-}(k/n)\right| &= |\sum_{\lambda}\sum_{\theta\leq\lambda}\lambda(D_n^{\lambda,\theta}(k)/n-d^{\lambda,\theta}(k/n))| \\ &\leq \sum_{\lambda}\sum_{\theta\leq\lambda}\lambda\left|D_n^{\lambda,\theta}(k)/n-d^{\lambda,\theta}(k/n)\right| \qquad \text{(I.35)}\end{aligned}$$

and

$$|D_n(k)/n-d(k/n)| \leq \sum_{\lambda}\sum_{\theta\leq\lambda}\left|D_n^{\lambda,\theta}(k)/n-d^{\lambda,\theta}(k/n)\right|, \qquad \text{(I.36)}$$

We obtain by using (I.29), for $0\leq k\leq n\hat{\tau}$ and n large:

$$\sup_{k\leq\hat{\tau}n}\left|D_n^{-}(k)/n-d^{-}(k/n)\right|\leq C\epsilon+o_p(1)$$
$$\sup_{k\leq\hat{\tau}n}|D_n(k)/n-d(k/n)|\leq C\epsilon+o_p(1)$$

We now study the stopping time T_n^{stop} defined in (I.19) and the size of the default cascade $D_n(T_n^{stop})$.

First assume $J(p)>p$ for all $p\in[0,1)$, i.e., $p^*=1$. Then we have

$$\forall t<\hat{t}, d^{-}(t)=\sum_{\lambda,\theta}\lambda d^{\lambda,\theta}(t)-t>0.$$

We have then that $T_n^{stop}/n=\hat{\tau}+O(\epsilon)+o_p(1)$ and from convergence (I.37), since $\delta(\hat{\tau})=1-O(\epsilon)$, we obtain by tending ϵ to 0 that $|D_n(T_n^{stop})|=n-o_p(n)$. This proves the first claim of the theorem.

Now consider the case when $p^*<1$, and furthermore p^* is a stable fixed point of $J(p)$. Then by definition of p^* and by using the fact that $I(1)\leq 1$, we have $I(p)<p$ for some interval $(p^*,p^*+\tilde{p})$. Then $d^{-}(t)$ is negative in an interval $(t^*,t^{stop}+\tilde{t})$, with $t^*=\bar{\lambda}p^*$.

Let ϵ such that $2\epsilon<-\inf_{t\in(t^*,t^{stop}+\tilde{t})}d^{-}(t)$ and denote $\hat{\sigma}$ the first iteration at which it reaches the minimum. Since $d^{-}(\hat{\sigma})<-2\epsilon$ it follows that with high probability $D^{-}(\hat{\sigma}n)/n<0$, so $T_n^{stop}/n=\tau^*+O(\epsilon)+o_p(1)$. The second claim of the theorem follows by taking the limit $\epsilon\to 0$.

Lemma 6.4 (Theorem 5.1. in [18]). *Let $b\geq 2$ be an integer and consider a sequence of real valued random variables $(\{Y_n^l(t)\}_{1\leq l\leq b})_{t\geq 0}$ and its natural filtration $\mathcal{F}_k^n$. Assume that there is a constant $C_0>0$ such that $|Y_n^l(k)|\leq C_0 n$ for all n, $t\geq 0$ and $1\leq l\leq b$. For all $l\geq 1$ let*

$f_l : \mathbb{R}^{b+1} \to \mathbb{R}$ be functions and assume that for some bounded connected open set $U \subseteq \mathbb{R}^{b+1}$ containing the closure of

$$\{(0, z_1, ..., z_b) : \exists\ n \text{ such that } (\forall\ 1 \leq l \leq b,\ Y_n^l(0) = z_l n) \neq 0\},$$

the following three conditions are verified:

(i) (Boundedness). *For some function $\beta(n) \geq 1$ we have for all $t < T_U$*

$$\max_{1 \leq l \leq b} |Y_n^l(t+1) - Y_n^l(t)| \leq \beta(n).$$

(ii) (Trend). *There exists $\lambda_1(n) = o(1)$ such that for $1 \leq l \leq b$ and $t < T_U$*

$$|\mathbb{E}[Y_n^l(t+1) - Y_n^l(t)|\mathcal{F}_{n,t}] - f_l(t/n, Y_n^1(t)/n, ..., Y_n^l(t)/n)| \leq \gamma_1(n).$$

(iii) (Lipschitz). *The functions $(f_l)_{1 \leq l \leq b}$ are Lipschitz-continuous on U.*

Then the following conclusions hold:

(a) *For $(0, \hat{z}_1, ..., \hat{z}_b) \in U$, the system of differential equations*

$$\frac{dz_l}{ds} = f_l(s, z_1, ..., z_l), \quad l = 1, ..., b,$$

has a unique solution in U, $z_l : \mathbb{R} \to \mathbb{R}$, which passes through $z_l(0) = \hat{z}_l$, for $l = 1, \ldots, b$, and which extends to points arbitrarily close to the boundary of U.

(b) *Let $\gamma > \gamma_1(n)$ with $\gamma = o(1)$. For a sufficiently large constant C, with probability $1 - O\left(\frac{b\beta(n)}{\gamma} \exp\left(-\frac{n\gamma^3}{\beta(n)^3}\right)\right)$, we have*

$$\sup_{0 \leq t \leq \sigma(n)n} (Y_n^l(t) - n z_n^l(t/n)) = O(\gamma n),$$

where $\mathbf{z}_n(t) = (z_n^1(t), \ldots, z_n^b(t))$ is the solution of

$$\frac{d\mathbf{z}_n}{dt} = f(t, \mathbf{z}_n(t)) \qquad \mathbf{z}_n(0) = \mathbf{Y}_n(0)/n$$

$$\text{and} \qquad \sigma(n) = \sup\{t \geq 0, \quad d_\infty(\mathbf{z}_n(t), \partial U) \geq C\gamma\}.$$

Chapter II

Extensions : Different recovery intensities and policies

In the previous chapter, we have analyzed systemic risk and network connectivity when nodes' threshold grows uniformly over time and thus have a uniform recivery. In this chapter, we propose several avenues of research on the optimal attribution policies for features over time and nodes.

1 Different recovery intensities modeling between two interactions

In the previous section we mentioned that since the number of links scales linearly with n then the time between interactions must scale with $\frac{1}{n}$. More precisely, we have investigated the case under assumption that the duration in calendar time between the two successive interactions follows an exponential distribution with parameter n, i.e. $\Delta_k = T_k^n - T_{k-1}^n \sim Exp(n)$. In order to allow the recovery intensity vary along time, we can extend this assumption by allowing the intensity of exponential distribution to be a general function of parameter n. This gives us the flexibility to control different recovery speed depends on the process of or the state of the cascade. One interesting example could be the case that the duration between two interactions starts to decrease as more and more unrevealed default links are added while the duration increases when the total number unrevealed defaulted links start to reduce indicating a higher interaction and recovery rate during the stage with more accumulated defaulted links. The former stage may corresponding to the early stage of the cascade process while the later case may corresponding to the later stage.

To be more precise, we can assume $\Delta_k = T_k^n - T_{k-1}^n \sim Exp(F_n(k))$, where $F_n(t)$ satisfying that $\lim_{n\to\infty} \frac{F_n(t)}{n} = f(\frac{t}{n})$ for some integrable function f on $[0, 1]$.

Apparently we have

$$\mathbb{E}(T_k^n) = \sum_{i=1}^{k} \frac{1}{F_n(i)} = \sum_{i=1}^{k} (\frac{i}{n} - \frac{i-1}{n}) \frac{1}{\frac{F_n(i)}{n}} \sim \int_0^{\frac{k}{n}} \frac{1}{f(s)} ds + o(\frac{1}{n})$$

$$Var(T_k^n) = \sum_{i=1}^{k} \frac{1}{F_n(i)} = \frac{1}{n} \sum_{i=1}^{k} (\frac{i}{n} - \frac{i-1}{n}) \frac{1}{(\frac{F_n(i)}{n})^2} \sim \int_0^{\frac{k}{n}} \frac{1}{f^2(s)} ds + o(\frac{1}{n}) \sim o(\frac{1}{n})$$

Thus again by Chebysev's inequality, we have as $n \to \infty$, in probability

$$T_k^n \xrightarrow{p} \int_0^{\frac{k}{n}} \frac{1}{f^2(s)} ds$$

The remaining part of the system keeps unchanged with the only replacement of definition $t_{min}^{\theta,l} = \inf\{t | l \leq \lambda \wedge (\theta + \frac{\alpha\lambda}{\lambda} t)\}$ to $t_{min}^{\theta,l} = \inf\{t | l \leq \lambda \wedge (\theta + \frac{\alpha\lambda}{\lambda} \int_0^t \frac{1}{f(s)} ds)\}$.

We now give a more concrete example where the intensity between the two successive interactions depends on how many unrevealed links in the system, i.e. $\Delta_k = T_k^n - T_{k-1}^n \sim Exp(D_n^-(k))$, where $D_n^-(k)$ the number of unrevealed links belonging to defaulted banks at time T_k^n as defined in (I.16). Following the arguments above we have similarly $\mathbb{E}(T_k^n) \sim \int_0^{\frac{k}{n}} \frac{1}{d^-(s)} ds + o(\frac{1}{n})$ and $Var(T_k^n) \sim \int_0^{\frac{k}{n}} \frac{1}{(d^-(s))^2} ds + o(\frac{1}{n}) \sim o(\frac{1}{n})$ and $t_{min}^{\theta,l} = \inf\{t | l \leq \lambda \wedge (\theta + \frac{\alpha\lambda}{\lambda} \int_0^t \frac{1}{d^-(s)} ds)\}$.

2 Different growth attribution policies

In the previous we have investigated the case when growth is linear in time and the total growth is distributed among nodes proportionally to their number of links. In this section we investigate two different types of growth features. Instead of allowing growth to be continuous in time, the first model in this section model the growth in the same fashion as the default : We incorporate both default and growth jumps into Markov jump diffusion processes with different jump intensities. At each interaction where the jumps happens, the jump is realized by a default or a growth according to the likelihood the proportional to connectivity and the number of the unrevealed links. This setup brings the advantages of unifying the modeling of both default and growth progresses compared to the previous model where continuous time growths need to be incorporated with the pure jump processes of default contagion. In the second type of model, we introduce a different type of growth distribution policy. Instead of uniformly distributed to all the available node, the growth capacity can be used only on the node with the most defaulted capital given their linkage and threshold. The growth capacity is thus used at best, on the nodes that, potentially, are the most likely to be default. This type of policy is of interests from a central party, for example a regulator or government's point of view whose aim seeks to minimize contagion.

2.1 Growth Policy I

In this section we investigate another two types of growth. We assume during each interaction time, either a default contagion or recovery happens with relative rate μ and μ^1. If the default happens, each default link has equal probability to be the undefaulted link. Moreover, if the growths happens, the institution gain the growth with the probability proportional to its connectivity (i.e. each link has the same probability to make the growth condition on the growth happens).

In the financial network of size n, given (l, k, θ) such that $0 \leq l \leq j$, $0 \leq k \leq d$ and $\theta \leq j$, we define $S_n^{\theta,j,k,l}(t)$ represent the number of banks with initial threshold θ, j connectivity, k units of growth and l default links at time t. The conservation relation

$$\sum_{0 \leq l \leq j, 0 \leq k \leq d} S_n^{\theta,j,k,l}(t) = S_n^{\theta,j,0,0}(0)$$

gives that the Markov processes $(S_n^{\theta,j,k,l}(t))_{0\le l\le j, 0\le k\le d}$ has the Q-matrix

$$\begin{cases} q^n(x, x - e_{(k,l)} + e_{(k,l+1)}) = \mu(j-l)x_{(k,l)} & 0\le k\le d, 0\le l\le j \\ q^n(x, x - e_{(k,l)} + e_{(k+1,l)}) = \mu^1 j x_{(k,l)} & \theta + k - l > 0, 0\le k\le d-1, 0\le l\le j. \\ q^n(x, x - e_{(d,l)}) = \mu^1 j x_{(d,l)} & \theta + d - l > 0, 0\le l\le j. \\ q^n(x, x - e_{(k,j)}) = \mu^1 j x_{(k,j)} & \theta + k - j > 0, 0\le k\le d. \end{cases} \tag{II.1}$$

For any RCLL function on $\mathbb{R}$, one denotes by $\mathcal{N}_h$ a point processes defined as follows

$$\mathcal{N}_h = \int_0^t \mathcal{P}([0, h(s-)] \times ds)$$

where $\mathcal{P}$ is a Poisson processes in $\mathbb{R}_+^2$ whose intensity is the Lebesgue measure on $\mathbb{R}_+^2$. Especially when h is deterministic $\mathcal{N}_h$ is Poisson processes with intensity $(h(t-))$.
Now we can represent the Markov processes $(S_n^{\theta,j,k,l}(t))_{0\le l\le j, 0\le k\le d}$ in terms of jump diffusion

$$\begin{cases} dS^{\theta,j,k,0}(t) = \mathcal{N}_{\mu_1 j S^{\theta,j,k-1,0}}(dt) - \mathcal{N}_{\mu_1 j S^{\theta,j,k,0}}(dt) - \mathcal{N}_{\mu(j-l)S^{\theta,j,k,0}}(dt) & \\ dS^{\theta,j,0,l}(t) = -\mathcal{N}_{\mu_1 j S^{\theta,j,0,l}}(dt) + \mathcal{N}_{\mu(j-l+1)S^{\theta,j,0,l-1}}(dt) - \mathcal{N}_{\mu(j-l)S^{\theta,j,0,l}}(dt) & l < \theta \\ dS^{\theta,j,0,l}(t) = \mathcal{N}_{\mu(j-l+1)S^{\theta,j,0,l-1}}(dt) - \mathcal{N}_{\mu(j-l)S^{\theta,j,0,l}}(dt) & l \ge \theta \end{cases} \tag{II.2}$$

$$\begin{cases} dS_N^{\theta,j,k,l}(t) = \mathcal{N}_{\mu^1 j S_N^{\theta,j,k-1,l}}(dt) - \mathcal{N}_{\mu^1 j S_N^{\theta,j,k,l}}(dt) + \mathcal{N}_{\mu(j-l+1)S_N^{\theta,j,k,l-1}}(dt) - \mathcal{N}_{\mu(j-l)S_N^{\theta,j,k,l}}(dt) & \theta + k - l > 0 \\ dS_N^{\theta,j,k,l}(t) = \mathcal{N}_{\mu(j-l+1)S_N^{\theta,j,k,l-1}}(dt) - \mathcal{N}_{\mu(j-l)S_N^{\theta,j,k,l}}(dt) & \theta + k - l \le 0 \end{cases} \tag{II.3}$$

Now let $U_N^{\theta,j,k,l}(t) = \int_0^t [\mathcal{N}_{\mu^1 j S_N^{\theta,j,k-1,l}}(du) - \mu^1 j S_N^{\theta,j,k-1,l} du] - \int_0^t [\mathcal{N}_{\mu_1 j S_N^{\theta,j,k,l}}(du) - \mu^1 j S_N^{\theta,j,k,l} du] + \int_0^t [\mathcal{N}_{\mu(j-l+1)S_N^{\theta,j,k,l-1}}(du) - \mu(j-l+1)S_N^{\theta,j,k,l-1} du] - \int_0^t [\mathcal{N}_{\mu(j-l)S_N^{\theta,j,k,l}}(du) - \mu(j-l)S_N^{\theta,j,k,l} du]$ be the martingales associated to the jumps of these processes.
And refer

$$W_N^{\theta,j,k,l}(t) = \int_0^t \mu^1 j S_N^{\theta,j,k-1,l} du - \int_0^t \mu^1 j S_N^{\theta,j,k,l} du + \int_0^t \mu(j-l+1) S_N^{\theta,j,k,l-1} du - \int_0^t \mu(j-l) S_N^{\theta,j,k,l} du$$

to the increasing part.

In the following we show that when $N \to \infty$ the sequence of processes

$$\Big(\frac{S_N^{\theta,j,k,l}(t)}{N}, 0\le l\le j, 0\le k\le j, \theta \le j\Big)$$

is tight and converges in distribution uniformly on compact sets to a continuous processes $(s^{\theta,j,k,l}(t))$. See Billingsley [?] for example.

Since $0 \le S_N^{\theta,j,k,l-1} \le N$, the above relation gives the existence of a constant C_1 such that

$$\mathbb{E}\big(U_N^{\theta,j,k,l}(t)^2\big) = \mathbb{E}\big(\langle U_N^{\theta,j,k,l}(t)\rangle(t)\big) \le C_1 N t$$

We can apply Doob's Inequality and gets that

$$\mathbb{P}\big(\sup_{0\le s\le t}\frac{U_N^{\theta,j,k,l}(s)}{N}\ge\epsilon\big)\le\frac{1}{(\epsilon N)^2}\mathbb{E}\big(U_N^{\theta,j,k,l}((t)^2\big)\le\frac{C_1 t}{\epsilon^2 N} \tag{II.4}$$

Which shows that the martingale $\big(\frac{U_N^{\theta,j,k,l}(s)}{N}\big)$ converges in probability to 0 uniformly on compact sets.
Tightness: For $T>0,\delta>0$,define $\omega_Z(\delta)$ as the modulus of continuity of the cadlag functions on the interval $[0,T]$.

$$\omega_Z(\delta)=\sup_{0\le s\le t\le T,|t-s|\le\delta}|Z(t)-Z(s)|$$

Then for any $\epsilon>0$ and $\eta>0$, there exists $\delta>0$ such that $\mathbb{P}(\omega_{S_N^{\theta,j,k,l}/N}(\delta)\ge\eta)\le\epsilon$.
Recall that $\frac{S_N^{\theta,j,k,l}}{N}=\frac{U_N^{\theta,j,k,l}}{N}+\frac{W_N^{\theta,j,k,l}}{N}$ (corresponds to the martingale part and the increasing part of $\frac{S_N^{\theta,j,k,l}}{N}$. Thus we have

$$\mathbb{P}(\omega_{S_N^{\theta,j,k,l}/N}(\delta)\ge\eta)\le\mathbb{P}(\omega_{U_N^{\theta,j,k,l}/N}(\delta)\ge\frac{\eta}{2})+\mathbb{P}(\omega_{W_N^{\theta,j,k,l}/N}(\delta)\ge\frac{\eta}{2})$$

The results follow by the inequality $\mathbb{P}(\omega_{U_N^{\theta,j,k,l}}(\delta)\ge\frac{\eta}{2})\le\mathbb{P}(\sup_{0\le s\le t}|\frac{U_N^{\theta,j,k,l}(s)}{N}|\ge\frac{\eta}{4})$.

The limit processes $(s^{\theta,j,k,l}(t))$ satisfies

$$\begin{cases} ds^{\theta,j,k,0}(t)=\mu^1(js^{\theta,j,k-1,0}-js^{\theta,j,k,0})(dt)+\mu\{-(j-l)s^{\theta,j,k,0})\}(dt) & \\ ds^{\theta,j,0,l}(t)=\mu^1(-js^{\theta,j,0,l})(dt)+\mu\{(j-l+1)s^{\theta,j,0,l-1}-(j-l)s^{\theta,j,0,l})\}(dt) & l<\theta \\ ds^{\theta,j,0,l}(t)=\mu(j-l+1)s^{\theta,j,0,l-1}(dt)-\mu(j-l)s^{\theta,j,0,l}(dt) & l\ge\theta \end{cases} \tag{II.5}$$

$$\begin{cases} ds^{\theta,j,k,l}(t)=\mu^1(js^{\theta,j,k-1,l}-js^{\theta,j,k,l})(dt)+\mu\{(j-l+1)s^{\theta,j,k,l-1}-(j-l)s^{\theta,j,k,l})\}(dt) & \theta+k-l>0 \\ ds^{\theta,j,k,l}(t)=\mu(j-l+1)s^{\theta,j,k,l-1}(dt)-\mu(j-l)s^{\theta,j,k,l}(dt) & \theta+k-l\le 0 \end{cases} \tag{II.6}$$

This can be rewrite as

$$\begin{cases} ds^{\theta,j,k,l}(t)=\mu^1js^{\theta,j,k-1,l}(dt)+\mu((j-l+1)s^{\theta,j,k,l-1}(dt)-\{\mu^1j+\mu(j-l)\}s^{\theta,j,k,l}(dt) & \theta+k-l>0 \\ ds^{\theta,j,k,l}(t)=\mu(j-l+1)s^{\theta,j,k,l-1}(dt)-\mu(j-l)s^{\theta,j,k,l}(dt) & \theta+k-l\le 0 \end{cases} \tag{II.7}$$

Then we have that

$$\begin{cases} \frac{d}{dt}(s^{\theta,j,k,l}e^{(\mu^1j+\mu(j-l))t})=\mu^1js^{\theta,j,k-1,l}+\mu(j-l+1)s^{\theta,j,k,l-1}\}e^{(\mu^1j+\mu(j-l))t} & \theta+k-l>0 \\ \frac{d}{dt}(s^{\theta,j,k,l}e^{(\mu(j-l))t})=\mu(j-l+1)s^{\theta,j,k,l-1}e^{\mu(j-l)t} & \theta+k-l\le 0 \end{cases} \tag{II.8}$$

Then we can solve the ODE system by recursion.

if $\theta + k - l > 0$

$$s^{\theta,j,k,l}(t) = e^{-(\mu^1 j + \mu(j-l))t} \int_0^t \{\mu^1 j s^{\theta,j,k-1,l}(u) + \mu(j-l+1)s^{\theta,j,k,l-1}(u)\} e^{(\mu^1 j + \mu(j-l))u} du$$

if $\theta + k - l \leq 0$

$$s^{\theta,j,k,l}(t) = e^{-\mu(j-l)t} \int_0^t \{\mu(j-l+1)s^{\theta,j,k,l-1}(u)\} e^{(\mu^1 j + \mu(j-l))u} du$$

Recall in the financial network of size n the definition of

$$\begin{aligned} D_n^{j,\theta}(t) &= n\mu_n(j,\theta) - \sum_{0 \leq l < \theta + k} S_n^{j,\theta,k,l}(t), \\ D_n^-(t) &= \sum_{j,\theta} j \cdot D_n^{j,\theta}(t) - \mathcal{N}_t^n, \\ D_n(t) &= \sum_{j,\theta} D_n^{j,\theta}(t) \end{aligned}$$

Where $\mathcal{N}_t^n$ is the jump diffusion processes to the link- reveal process. They are generalized Poisson processes with intensity $\int_0^t \lambda_s^n ds$ with $\lambda_t^n = \mu(m_n - \mathcal{N}_t^n)$

Informally, we have

$$\frac{\mathcal{N}_t^n}{n} = \frac{\int_0^t \lambda_s^n ds}{n} + \frac{\mathcal{N}_t - \int_0^t \lambda_s^n dt}{n} \tag{II.9}$$

Let $\frac{\hat{\mathcal{N}}_t}{n}$ be the martingale $\frac{\mathcal{N}_t - \int_0^t \lambda_s^n ds}{n}$, then as before it convergences in distribution to 0. Thus if we let $\kappa_t = lim_{n \to \infty} \frac{\mathcal{N}_t^n}{n}$. Then from equation (1.1),κ_t satisfies

$$\kappa_t = \mu(\lambda t - \int_0^t \kappa_s ds) \tag{II.10}$$

which gives $\kappa_t = \lambda(1 - e^{-\mu t})$.

The length of the default cascade is given by

$$T_n^{stopp} = \inf\{0 \leq t \leq \infty,\ D_n^-(t) = 0\} \tag{II.11}$$

Since both $D_n(t)$ and $\mathcal{N}(t)$ takes value in $\mathbb{N}$, the above is equivalent to

$$T_n^{stop} = \inf\{\tau_i^n,\ 0 \leq i \leq m_n,\ D_n^-(\tau_i) = 0\} \tag{II.12}$$

Where $\tau_i^n = \inf\{0 < t < \infty, \mathcal{N}_t^n = i\}$. And we are interested in the proportion $\frac{D(T_n^{stop})}{n}$. In the limiting case ($n \to \infty$), it is equivalent to find

$$t^{stop} = \inf\{0 \leq t \leq \infty, \sum_{j,\theta} j \cdot d^{j,\theta}(t) = \lambda(1 - e^{-\mu t})\}$$

And the finally default proportion in limit case is given by $d(t^{stop})$.

2.2 Growth Policy II

In the second policy, we assume at each round of growth, the maximal capacity of growth is $\mu_1 n$ (with maximal capacity $\mu_1 S_n^{\theta,j,0,0}(0)$ for banks with threshold θ and connectivity j).
The growth capacity can be used only on the banks with the *most defaulted capital* given their linkage and threshold. The growth capacity is in fact used at best, on the banks that, potentially, are the most likely to be lost.
Let $X_n^{\theta,j,l}(t)$ be the number of non-defaulted banks with capacity θ, j outgoing links and l "net" defaulted links (=the defaulted links−the recovered links) at time t. We assume that the defaulted links can recover according to the above Growth Policy II. Given θ and j, the Q matrix will be

$$\begin{cases} q^n(x, x-e_l+e_{l+1}) = \mu(j-l)x_l & 0 \le l \le j \\ q^n(x, x+e_l-e_{l+1}) = \mu^1 n\mu_n(j,\theta)1_{\{x_{l+1}>0, x_i=0, l+2\le i\le j\}} & l<\theta, 0\le l\le j-2. \end{cases} \tag{II.13}$$

$$\begin{cases} dX_n^{\theta,j,0}(t) = 1_{\{X_n^{\theta,j,1}>0, X_n^{\theta,j,i}=0, 2\le i\le j\}}\mathcal{N}_{\mu^1 n\mu_n(j,\theta)}(dt) - \mathcal{N}_{\mu j X_n^{\theta,j,0}}(dt) \\ dX_n^{\theta,j,l}(t) = 1_{\{X_n^{\theta,j,l+1}>0, X_n^{\theta,j,i}=0, l+2\le i\le j\}}\mathcal{N}_{\mu^1 n\mu_n(j,\theta)}(dt) - 1_{\{X_n^{\theta,j,l}>0, X_n^{\theta,j,i}=0, l+1\le i\le j\}}\mathcal{N}_{\mu^1 n\mu_n(j,\theta)}(dt) \\ +\mathcal{N}_{\mu(j-\theta+1)X_n^{\theta,j,l-1}}(dt) - \mathcal{N}_{\mu(j-l)X_n^{\theta,j,l}}(dt) & 1\le l\le \theta-2 \\ dX_n^{\theta,j,\theta-1}(t) = -1_{\{X_n^{\theta,j,\theta-1}>0, X_n^{\theta,j,i}=0, \theta\le i\le j\}}\mathcal{N}_{\mu^1 n\mu_n(j,\theta)}(dt) + \mathcal{N}_{\mu(j-\theta+2)X_n^{\theta,j,\theta-2}}(dt) \\ -\mathcal{N}_{\mu(j-\theta+1)X_n^{\theta,j,\theta-1}}(dt) & l=\theta-1 \\ dX_n^{\theta,j,l}(t) = \mathcal{N}_{\mu(j-\theta+1)X_n^{\theta,j,l-1}}(dt) - \mathcal{N}_{\mu(j-l)X_n^{\theta,j,l}}(dt) & \theta\le l\le j \end{cases} \tag{II.14}$$

$$\begin{cases} dX_n^{\theta,j,0}(t) = 1_{\{X_n^{\theta,j,1}>0, X_n^{\theta,j,i}=0, 2\le i\le j\}}\mu^1 n\mu_n(j,\theta)dt - \mu j X_n^{\theta,j,0}dt + M_n^{\theta,j,0}(t) \\ dX_n^{\theta,j,l}(t) = \mu^1 n\mu_n(j,\theta)1_{\{X_n^{\theta,j,l+1}>0, X_n^{\theta,j,i}=0, l+2\le i\le j\}}(dt) - \mu^1 n\mu_n(j,\theta)1_{\{X_n^{\theta,j,l}>0, X_n^{\theta,j,i}=0, l+1\le i\le j\}}(dt) \\ +\mu\{(j-\theta+1)X_n^{\theta,j,l-1} - (j-l)X_n^{\theta,j,l}\}(dt) + M_n^{\theta,j,l}(t) & 1\le l\le\theta-2 \\ dX_n^{\theta,j,l}(t) = -\mu^1 n\mu_n(j,\theta)1_{\{X_n^{\theta,j,l}>0, X_n^{\theta,j,i}=0, l+1\le i\le j\}}(dt) \\ +\mu\{(j-\theta+1)X_n^{\theta,j,l-1} - (j-l)X_n^{\theta,j,l}\}(dt) + M_n^{\theta,j,l}(t) & l=\theta-1 \\ dX_n^{\theta,j,l}(t) = \mu\{(j-\theta+1)X_n^{\theta,j,l-1} - (j-l)X_n^{\theta,j,l})\}(dt) + M_n^{\theta,j,l}(t) & \theta\le l\le j \end{cases} \tag{II.15}$$

Now Let $R_n^{\theta,j,l}(t)$ the number of banks with more than l defaulted links, j connectivity, θ threshold at time t and $I_n^{\theta,j,l}(t)$ is the local time at 0 of $R_n^{\theta,j,l}(t)$.
$R_n^{\theta,j,l}(t) = \sum_{l\le i\le\theta-2} X_n^{\theta,j,i}(t)$ for $l\le\theta-2$ and $I_n^{\theta,j,l}(t) = \int_0^t 1_{\{R_n^{\theta,j,l}(u)=0\}}du$. We also make the convention that $R_n^{\theta,j,\theta-1}(t)=0$ and $I_n^{\theta,j,\theta-1}(t)=t$.
Then the condition $\{X_n^{\theta,j,l}>0, X_n^{\theta,j,i}=0, l+1\le i\le j\}$ is equivalent to $\{R_n^{\theta,j,l}>0, R_n^{\theta,j,l+1}=0\}$. The evolution of $R_n^{\theta,j,l}(t)$ can then be written as

$$\begin{cases} dR_n^{\theta,j,0}(t) = -\mathcal{N}_{\mu(j-\theta+2)X_n^{\theta,j,\theta-2}}(dt) \\ dR_n^{\theta,j,l}(t) = -1_{\{R_n^{\theta,j,l}>0, R_n^{\theta,j,l+1}=0\}}\mathcal{N}_{\mu^1 n\mu_n(j,\theta)}(dt) \\ +\mathcal{N}_{\mu(j-l+1)X_n^{\theta,j,l-1}}(dt) - \mathcal{N}_{\mu(j-\theta+2)X_n^{\theta,j,\theta-2}}(dt) \qquad 1\le l\le\theta-2 \end{cases} \tag{II.16}$$

By compensating the jump diffusion in the above equation, and notice that $X_n^{\theta,j,l}(t) = R_n^{\theta,j,l}(t) - R_n^{\theta,j,l+1}(t)$ and $X_n^{\theta,j,\theta-2}(t) = R_n^{\theta,j,\theta-2}(t)$.
Directly from (II.16), notice that $1_{\{R_n^{\theta,j,l}>0, R_n^{\theta,j,l+1}=0\}} = 1_{\{R_n^{\theta,j,l+1}=0\}} - 1_{\{R_n^{\theta,j,l}=0\}}$. One gets

$$R_n^{\theta,j,l}(t) = H_n^{\theta,j,l}(t) - \mu_1 n\mu_n(j,\theta)(I_n^{\theta,j,l+1}(t) - I_n^{\theta,j,l}(t)) \tag{II.17}$$

with

$$dH_n^{\theta,j,l}(t) = \mu(j-l+1)(R_n^{\theta,j,l} - R_n^{\theta,j,l-1})(dt) - \mu(j-\theta+2)R_n^{\theta,j,\theta-2}(dt) + dV_n^{\theta,j,l}(t) \tag{II.18}$$

Where $V_n^{\theta,j,l}(t)$ are the martingales associated to the jumps of the processes. Similar to the previous section,We can show that

$$\mathbb{P}\big(sup_{0\leq s\leq t}\frac{V_N^{\theta,j,k,l}(s)}{N} \geq \epsilon\big) \leq \frac{1}{(\epsilon N)^2}\mathbb{E}\big(V_N^{\theta,j,k,l}((t)^2\big) \leq \frac{C_1 t}{\epsilon^2 N}$$

Thus $\big(\frac{V_N^{\theta,j,k,l}(s)}{N}\big)$ converges in probability to 0 uniformly on compact sets.

Fix θ, j, Equation(II.17)-(II.18) can be interpreted as the fact that the couple $(R_n^{\theta,j}, \mu^1 n\mu_n(j,\theta)I_n^{\theta,j})$ is the solution of the Generalized Skorohod Problem (see the definition below, or see the Appendix D of Robert [?]) associated to the processes $(G_l(R_n^{\theta,j}) + V_{n,l})_l$. Where

$$G_l^{\theta,j}(r) = \mu\int_0^t \big((j-l+1)(r^l - r^{l-1}) - (j-\theta+2)r^{\theta-2}\big)du$$

And matrix $P = (P_{i,j})$ whose non zero coefficients are $P_{i,i+1} = 1$ for $0 \leq i \leq \theta - 2$ Thus in the limit case, $(\frac{R_n^{\theta,j,l}(t)}{n}, \frac{I_n^{\theta,j,l}(t)}{n})$ converges to the solution (r,i) of the Generalized Skorohod problem with Functional G and matrix P.As proved in Appendix D of Robert [?], we have the existence and uniqueness of the Generalized Skorohod problem. And in this case we can even explicitly solve the problem due to the linear topology of functional G is linear in r.

Having the solution of $R^l(t)$, we can then recover the solution of $X^l(t)$ from $R^l(t)$ (recall that $X^l(t) = R^l(t) - R^{l+1}(t)$).

Definition 2.1. *(Skorohod Problem).*
If $(Y(t))$ is a cadlag functions with values in $\mathbb{R}^d$ such that $Y(0) \leq 0$ and $P = (p_{ij})$ is a $d\times d$-matrix, a solution to Skorohod Problems associated with $(Y(t))$ and P is couple of functions $(X(t)) = (X_i(t); 1 \leq i \leq d)$ and $(I(t)) = (I_i(t); 1 \leq i \leq d)$ in $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$ such that for any $t \geq 0$,

- *$X(t) = Y(t) + (1 - P^T)I(t)$;*
- *for $1 \leq i \leq d$, $X_i(t) \geq 0$, the function $t \to I_i(t)$ is non decreasing and $I_i(0) = 0$;*
- *the reflection condition:*
$$\int_0^\infty X_i(s)dI_1(s) = 0$$

Definition 2.2. *(Generalized Skorohod Problem).*
If $G : \mathcal{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$ and $P = (p_{ij})$ is a $d\times d$- matrix, a solution to Generalized Skorohod Problems associated with G and P is couple of functions $(X(t)) = (X_i(t); 1 \leq i \leq d)$ and $(I(t)) = (I_i(t); 1 \leq i \leq d)$ in $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$ such that for any $t \geq 0$,

- $X(t) = G(X(t)) + (I - P^T)R(t)$;
- *for* $1 \leq i \leq d$, $X_i(t) \geq 0$, *the function* $t \to I_i(t)$ *is non decreasing and* $I_i(0) = 0$;
- *the reflection condition:*

$$\int_0^\infty X_i(s) dI_i(s) = 0$$

$$\begin{cases} dx^{\theta,j,l}(t) = \mu^1\mu(j,\theta)1_{\{x^{\theta,j,l+1}>0, x^{\theta,j,i}=0, l+2\leq i\leq j\}}(dt) - \mu^1\mu(j,\theta)1_{\{x^{\theta,j,l}>0, x^{\theta,j,i}=0, l+1\leq i\leq j\}}(dt) \\ +\mu\{(j-l+1)x^{\theta,j,l-1} - (j-l)x^{\theta,j,l}\}(dt) \end{cases} \quad 1 \leq l \leq \theta - 2 \tag{II.19}$$

Bibliography

[1] H. Amini, R. Cont, and A. Minca. Resilience to contagion in financial networks. *Mathematical Finance*, 26(2):329–365, 2016.

[2] H. Amini and A. Minca. Inhomogeneous financial networks and contagious links. *Operations Research*, 64(5):1109–1120, 2016.

[3] H. Amini, A. Minca, and A. Sulem. Control of interbank contagion under partial information. *SIAM Journal on Financial Mathematics*, 6(1):1195–1219, 2015.

[4] J. Blanchet and Y. Shi. Stochastic risk networks: Modeling, analysis and efficient Monte Carlo. 2012.

[5] L. Blume, D. Easley, J. Kleinberg, R. Kleinberg, and É. Tardos. Network formation in the presence of contagious risk. In *Proceedings of the 12th ACM conference on Electronic commerce*, pages 1–10. ACM, 2011.

[6] R. Carmona, J.-P. Fouque, and L.-H. Sun. Mean field games and systemic risk. *Communications in Mathematical Sciences*, 13(4):911–933, 2015.

[7] R. Chen, A. Minca, and A. Sulem. Optimal connectivity for a large financial network. *ESAIM: Proceedings and Surveys*, 59:43–55, 2017.

[8] S. Erol. Network hazard and bailouts. *Working paper*, 2015, rhsmith.umd.edu/files/Documents/Departments/Finance/seminars-spring2016/erol.pdf.

[9] S. Erol and R. Vohra. Network formation and systemic risk. *PIER Working paper*, (14-029), 2014, https://ssrn.com/abstract=2500846.

[10] J.-P. Fouque and T. Ichiba. Stability in a model of interbank lending. *SIAM Journal on Financial Mathematics*, 4(1):784–803, 2013.

[11] J. Garnier, G. Papanicolaou, and T.-W. Yang. Diversification in financial networks may increase systemic risk. *Handbook on Systemic Risk*, pages 432–444, 2013.

[12] J. Garnier, G. Papanicolaou, and T.-W. Yang. Large deviations for a mean field model of systemic risk. *SIAM Journal on Financial Mathematics*, 4(1):151–184, 2013.

[13] S. Janson. The probability that a random multigraph is simple. *Combinatorics, Probability and Computing*, 18(1-2):205–225, 2009.

[14] O. Kley, C. Klüppelberg, and G. Reinert. Risk in a large claims insurance market with bipartite graph structure. *Operations Research*, 64(5):1159–1176, 2016.

[15] A. Minca and A. Sulem. Optimal control of interbank contagion under complete information. *Statistics & Risk Modeling*, 31(1):23–48, 2014.

[16] L. C. Rogers and L. A. Veraart. Failure and rescue in an interbank network. *Management Science*, 59(4):882–898, 2013.

[17] T. Roukny, H. Bersini, H. Pirotte, G. Caldarelli, and S. Battiston. Default cascades in complex networks: Topology and systemic risk. *Scientific reports*, 3:2759, 2013.

[18] N. C. Wormald. Differential equations for random processes and random graphs. *The annals of applied probability*, pages 1217–1235, 1995.

Part 3

Optimization problems for Mean-Field BSDEs with jumps

CHAPTER III

Introduction

In this part, we begin by introducing two types of Mean-Field BSDEs and the corresponding Reflected Mean-Field BSDEs.

First-type of Mean-Field (R)BSDEs The first-type of Mean-field BSDE is a process $(X, Z, l) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ satisfying

$$\begin{cases} -dX_t = f(t, \omega, F(t, X_t(\cdot)), X_t, Z_t, l_t(\cdot))dt - Z_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt, de); \\ X_T = \xi. \end{cases} \tag{III.1}$$

where F is $\mathcal{B}([0,T]) \times \mathcal{B}(L^2_{\mathbf{P}})$ measurable operator from $[0,T] \times L^2_{\mathbf{P}}(\mathcal{F}_T)$ to $\mathbb{R}$.

Correspondingly the Mean-field Reflected BSDE of the first-type is a process $(Y, Z, k(.), A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$ satisfying

$$\begin{cases} -dY_t = f(t, \omega, F(t, X_t(\cdot)), X_t, Z_t, k_t(\cdot))dt + dA_t - Z_t dW_t - \int_U k_t(u)\tilde{N}(dt, du); \quad Y_T = \xi_T, \\ Y_t \geq \xi_t, \ 0 \leq t \leq T \text{ a.s.}, \\ A \text{ is a nondecreasing RCLL continous process with } A_0 = 0 \text{ and such that} \\ \int_0^T (Y_t - \xi_t) dA^c_t = 0 \text{ a.s. and } \ \Delta A^d_t = -\Delta Y_t \, \mathbf{1}_{\{Y_{t-} = \xi_{t-}\}} \quad \text{a.s.} \end{cases}$$

Second-type of Mean-Field (R)BSDEs The second-type of Mean-field BSDE is a process $(X, Z, l) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ satisfying

$$\begin{cases} -dX_t = \mathbb{E}'\left[f(t, \omega, X'_t, Z'_t, l'_t(\cdot), X_t, Z_t, l_t(\cdot))\right] dt - Z_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt, de); \\ X_T = \xi. \end{cases} \tag{III.2}$$

where $\mathbb{E}' f(t, \omega, X'_t, Z'_t, l'_t(\cdot), X_t, Z_t, l_t(\cdot)) = \int_\Omega f(t, \omega, X_t(\omega'), Z_t(\omega'), l_t(\cdot)(\omega'), X_t(\omega), Z_t(\omega), l_t(\cdot)(\omega)) d\mathbb{P}(\omega')$

Correspondingly the Mean-field Reflected BSDE of the second-type is a process $(Y, Z, k(.), A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$ satisfying

$$\begin{cases} -dY_t = \mathbb{E}'\left[f(t, \omega, X'_t, Z'_t, k'_t(\cdot), X_t, Z_t, k_t(\cdot))\right] dt + dA_t - Z_t dW_t - \int_U k_t(u)\tilde{N}(dt, du); \quad Y_T = \xi_T, \\ Y_t \geq \xi_t, \ 0 \leq t \leq T \text{ a.s.}, \\ A \text{ is a nondecreasing RCLL continous process with } A_0 = 0 \text{ and such that} \\ \int_0^T (Y_t - \xi_t) dA^c_t = 0 \text{ a.s. and } \ \Delta A^d_t = -\Delta Y_t \, \mathbf{1}_{\{Y_{t-} = \xi_{t-}\}} \quad \text{a.s.} \end{cases}$$

The second type of Mean-Field BSDE is originally introduced and studied by Buckdahn *et al*

[9] [10] and by Li *et al* [8] for the case with jumps. The solution of the second type Mean-field BSDE (III.2) was related to the solution (X^N, Z^N, l^N) of the backward equation

$$\begin{cases} -dX_t = \frac{1}{N}\sum_{j=1}^N \left[f(t,\omega,X_t^{j,N},Z_t^{j,N},l_t^{j,N}(\cdot),X_t,Z_t,l_t(\cdot))\right]dt - Z_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt,de); \\ X_T = \xi. \end{cases} \tag{III.3}$$

where the i.i.d. sequence $(X^{j,N}, Z^{j,N}, l^{j,N})$, $1 \le j \le N$, are following the same law as (X, Z, l). This convergence result is proved in [9] in the case of Brownian motion, namely they show (X, Z, l) to be the uniform limit of the solution (X^N, Z^N, l^N) when $N \to \infty$.

It is typical in the literature to motivate the study of the second type Mean-field BSDE III.2 by considering it as an approximation of a system of N coupled symmetric stochastic differential equations

$$\begin{cases} -dX_t^{i,N} = \frac{1}{N}\sum_{j=1}^N f(t,\omega,X_t^{j,N},X_t^{i,N})dt - Z_t^{i,N}dW_t - \int_{\mathbf{E}} l_t^{i,N}(e)\tilde{N}(dt,de); \\ X_T^{i,N} = \xi. \end{cases}$$

In contrast, the first type of Mean-Field BSDE was introduced only recently by Agram *et al* [14] with the specific form for the operator F as expectation: $F(t,X) = \mathbb{E}[\varphi(t,X)]$. In this simple case, the first type Mean-Field BSDE can be seen as a limit of the coupled system

$$\begin{cases} -dX_t^{i,N} = f(t,\omega,\frac{1}{N}\sum_{j=1}^N \varphi(X_t^{j,N}),X_t^{i,N},Z_t^{i,N},l_t^{i,N}(\cdot))dt - Z_t^{i,N}dW_t - \int_{\mathbf{E}} l_t^{i,N}(e)\tilde{N}(dt,de); \\ X_T^{i,N} = \xi. \end{cases}$$

Both types of N coupled equations are grounded in mean field games, depending what kind of interactions among players we consider.

However, the operator F in the first type of mean field BSDEs (III.1) should not limited to the linear or order 1 type of interactions as in the example above, as we can incorporate higher orders of interactions.

For example, quadratic or order 2 interactions can be captured by the following operator

$$F(t,X_t) = \mathbb{E}'\left[\mathbb{E}\left[f(t,X_t,X_t^{'})\right]\right].$$

This can be seen as a mean field limit of the N- coupled system with drift interaction of the form

$$\frac{1}{N^2}\sum_{j,k} f(t,X_t^{i,N},X_t^{j,N},X_t^{k,N}).$$

This motivate us to study in more detail the first-type Mean-Field BSDE with generalized operator F and the related Reflected BSDEs. Along the way, we only sketch the results for the Second-type BSDSs, much more pervasively studied in the literature.

For both type of Mean-Field BSDE, the solution to the corresponding Reflected BSDEs is related through the optimal stopping problem

$$Y = \operatorname*{ess\,sup}_{\tau\in\mathbb{T}} X(\xi_\tau,\tau) \quad \text{a.s.} \tag{III.4}$$

The purpose of later sections are the following.

(i) To prove (strict) comparison results for the above first type of Mean-field BSDE.

(ii) To apply the obtained results to dynamic risk measure and give a dual represent results.

(iii) To prove existence and comparison results for Reflected Mean-field BSDE.

(iv) To prove a optimal stopping relation between the mean field BSDE and its reflected counterpart with the application to optimal stopping problem of dynmaic risk measure.

(v) To show the optimization principle mean-field (R)BSDE as well as results of robust optimal stopping problems.

(vi) Sketch the corresponding results and proofs for the second-type mean-field (R)BSDE.

Chapter IV

Mean-field BSDE and application to Global Dynamic Risk Measures

---

1 Mean-field BSDEs with jumps

1.1 Notation and definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let W be a one-dimensional Brownian motion. Let $\mathbf{E} := \mathbb{R}^*$ and $\mathcal{B}(\mathbf{E})$ be its Borelian filtration. Suppose that it is equipped with a σ-finite positive measure ν and let $N(dt, de)$ be a Poisson random measure with compensator $\nu(de)dt$. Let $\tilde{N}(dt, de)$ be its compensated process. Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the completed natural filtration associated with W and N.

Notation. Let $\mathcal{P}$ be the predictable σ-algebra on $[0, T] \times \Omega$.

For each $T > 0$, we use the following notation: $L^2(\mathcal{F}_T)$ is the set of random variables ξ which are $\mathcal{F}_T$-measurable and square integrable; $\mathbb{H}^2$ is the set of real-valued predictable processes ϕ such that $\|\phi\|^2_{\mathbb{H}^2} := E\left[\int_0^T \phi_t^2 dt\right] < \infty$; $\mathcal{S}^2$ denotes the set of real-valued RCLL adapted processes ϕ such that $\|\phi\|^2_{\mathcal{S}^2} := E(\sup_{0 \leq t \leq T} |\phi_t|^2) < \infty$; $\mathcal{A}^2$ (resp. $\mathcal{A}^1$) is the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $E(A_T^2) < \infty$ (resp. $E(A_T) < \infty$). We also introduce the following spaces:

- L^2_ν is the set of Borelian functions $\ell : \mathbf{E} \to \mathbb{R}$ such that $\int_{\mathbf{E}} |\ell(e)|^2 \nu(de) < +\infty$.
  The set L^2_ν is a Hilbert space equipped with the scalar product
  $\langle \ell, \ell' \rangle_\nu := \int_{\mathbf{E}} \ell(e)\ell'(e)\nu(de)$ for all $\ell, \ell' \in L^2_\nu \times L^2_\nu$, and the norm $\|\ell\|^2_\nu := \int_{\mathbf{E}} |\ell(e)|^2 \nu(de)$.
- $\mathbb{H}^2_\nu$ is the set of all mappings $l : [0, T] \times \Omega \times \mathbf{E} \to \mathbb{R}$ that are $\mathcal{P} \otimes \mathcal{B}(\mathbf{E})/\mathcal{B}(\mathbb{R})$ measurable and satisfy $\|l\|^2_{\mathbb{H}^2_\nu} := E\left[\int_0^T \|l_t\|^2_\nu \, dt\right] < \infty$, where $l_t(\omega, e) = l(t, \omega, e)$ for all $(t, \omega, e) \in [0, T] \times \Omega \times \mathbf{E}$.

Moreover, $\mathbb{T}_0$ is the set of stopping times τ such that $\tau \in [0, T]$ a.s. and for each S in $\mathbb{T}_0$, we denote by $\mathbb{T}_S$ the set of stopping times τ such that $S \leq \tau \leq T$ a.s.

Definition 1.1. *([12] Definition 2.1) A progressive process (ϕ_t) is said to be left-upper semi-continuous (l.u.s.c.) along stopping times if for all $\tau \in \mathbb{T}_0$ and for each non decreasing sequence of stopping times (τ_n) such that $\tau^n \uparrow \tau$ a.s.,*

$$\phi_\tau \geq \limsup_{n \to \infty} \phi_{\tau_n} \quad \text{a.s.} \tag{IV.1}$$

Definition 1.2 (Driver, Lipschitz driver)**.** *A function f is said to be a* driver *if*

- $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^2 \times L^2_\nu \to \mathbb{R}$
  $(\omega, t, y', y, z, l(\cdot)) \mapsto f(\omega, t, y', y, z, l(\cdot))$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(L^2_\nu)-$ *measurable,*
- $f(., 0, 0, 0, 0, 0) \in \mathbb{H}^2$.

A driver f *is called a* Lipschitz driver *if moreover there exists a constant* $C \geq 0$ *such that* $dP \otimes dt$*-a.s., for each* (y'_1, y_1, z_1, l_1), (y'_2, y_2, z_2, l_2),

$$|f(\omega, t, y'_1, y_1, z_1, l_1) - f(\omega, t, y'_2, y_2, z_2, l_2)| \tag{IV.2}$$

$$\leq C(|y'_1 - y'_2| + |y_1 - y_2| + |z_1 - z_2| + \|l_1 - l_2\|_\nu). \tag{IV.3}$$

Definition 1.3. *A operator* F *on* $L^2_{\mathbf{P}}(\mathcal{F}_T)$ *is said to be an Mean-field operator* F *if*

- $F : [0,T] \times L^2_{\mathbf{P}}(\mathcal{F}_T) \to \mathbb{R}$
  $(t, X) \mapsto F(t, X)$ *is* $\mathcal{B}([0,T]) \times \mathcal{B}(L^2_{\mathbf{P}})-$ *measurable,*
- *For each* $t \in [0,T]$, $F(t, 0) < +\infty$.

An Mean-field operator is called a Lipschitz Mean-field operator *if there exists a a constant* $C \geq 0$*, such that for each* $(X_1, X_2) \in \mathcal{X} \times \mathcal{X}$

$$|F(t, X_1) - F(t, X_2)| \leq C\|X_1 - X_2\|_{L^2_{\mathbf{P}}} \tag{IV.4}$$

Remark 1.4. *An example is to take* $F(t, X) := \mathbb{E}[\phi(t, X)]$ *for* $X \in L^2_{\mathbf{P}}(\mathcal{F}_T)$ *, where*

$$\phi : [0,T] \times \mathbb{R} \mapsto \mathbb{R}, (t, x) \mapsto \phi(t, x)$$

is a concave Lipschitz function such that $\phi(t, X) \in L^2_{\mathbf{P}}(\mathcal{F}_T)$.

Definition 1.5 (Mean-field BSDE with jump)**.** *A solution of a Mean-field BSDE with jumps with terminal time* T*, terminal condition* ξ *and driver* f *and operator* F *consists of a triple of processes* $(X, Z, l) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ *satisfying*

$$-\, dX_t = f(t, \omega, F(t, X_t(\cdot)), X_t, Z_t, l_t(\cdot))dt - Z_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt, de); \tag{IV.5}$$

$$X_T = \xi.$$

where X *is a RCLL optional process, and* Z *(resp.* k*) is an* $\mathbb{R}$*-valued predictable process defined on* $\Omega \times [0,T]$ *(resp.* $\Omega \times [0,T] \times \mathbb{R}^*$*) such that the stochastic integral with respect to* W *(resp.* $\tilde{N}$*) is well defined. We denote by* $(X(\xi, T), Z(\xi, T), l(\xi, T))$ *the solution of the Mean-field BSDE associated with terminal time* T *and* (ξ, f).

We now give existence and uniqueness results, using a Banach fixed-point theorem, which takes into account the presence of the additional mean-field operator $\mathbf{F}$ and the careful tuning of the Lipschitz property of both f and $\mathbf{F}$.

Theorem 1.1. *(Existence and Uniqueness Mean-field BSDE)[14]*
Let f *be a Lipschitz driver and* $\mathbf{F}$ *a Mean-field operator. The Mean-field BSDE (IV.5) admits a unique solution* $(X, Z, \ell(.)) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$.

1.2 Comparison Results

In this section, in order to compare the first components of the solutions of two mean-field BSDEs, we need additional monotonicity assumptions due to the presence of jumps and of the mean-field operator.

Assumption 3.1. *Assume that $dP \otimes dt$-a.s for each $(x', x, z, \ell_1, \ell_2) \in \mathbb{R}^3 \times (L^2_\nu)^2$,*

$$f(t, x', x, z, \ell_1) - f(t, x', x, z, \ell_2) \geq \langle \theta_t^{x', x, z, \ell_1, \ell_2} \,,\, \ell_1 - \ell_2 \rangle_\nu,$$

with

$$\theta_t^{x', x, z, l_1, l_2} : [0, T] \times \Omega \times \mathbb{R}^3 \times (L^2_\nu)^2 \to L^2_\nu \,;\, (t, \omega, x', x, z, \ell_1, \ell_2) \mapsto \theta_t^{x', x, z, l_1, l_2}(\omega, .)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable, bounded, and satisfying $dP \otimes dt \otimes d\nu(u)$-a.s., for each $(x', x, z, \ell_1, \ell_2) \in \mathbb{R}^3 \times (L^2_\nu)^2$,

$$\theta_t^{x', x, z, l_1, l_2}(u) \geq -1 \quad \text{and} \quad |\theta_t^{x', x, z, l_1, l_2}(u)| \leq \psi(u), \tag{IV.6}$$

where $\psi \in L^2_\nu$.

Theorem 1.2 (Comparison Theorem for Mean-field BSDEs with jumps)**.** *Let $f_i = f_i(\omega, t, x', x, z, l)$,$i = 1, 2$, be two Lipschitz drivers, and one of them satisfy Assumption 3.1. Furthermore, we assume:*

- *One of the drivers f_i is nondecreasing in x';*

Let F be a Lipschtiz operator on $L^2(\Omega)$ satisfying the following property:

- *F is nondecreasing in x in the following sense: let $x_1, x_2 \in L^2(\Omega)$, if $x_1 \leq x_2$ a.s. , then for each $t \in [0, T]$, $F(t, x_1) \leq F(t, x_2)$.*

Let $\xi_1, \xi_2 \in L^2(\mathcal{F}_T)$ and denote by (X^1, Z^1, l^1) and (X^2, Z^2, l^2) the solution of the mean-field BSDE with jump (IV.5) associated with (ξ_1, f_1) and (ξ_2, f_2). Suppose that

- *$\xi_1 \geq \xi_2$, a.s.*
- *$f_1(\omega, t, x', x, z, l(\cdot)) \geq f_2(\omega, t, x', x, z, l(\cdot))$,a.s.*
  for all $(x', x, z, l(\cdot)) \in \mathbb{R}^2 \times L^2_\nu \times \mathbb{R}^2 \times L^2_\nu$

Then we have $X^1_t \geq X^2_t$, $\forall t \in [0, T]$ a.s.

Proof. For $i = 1, 2$, let $(X^{i,n}_s, Z^{i,n}_s, l^{i,n}_s)$ be the solution of the following iterating BSDE with jumps

$$X^{i,n}_s = \xi_i + \int_t^T f_i(s, F(s, X^{i,n-1}_s(\cdot)), X^{i,n}_s, Z^{i,n}_s, l^{i,n}_s)ds - \int_t^T Z^{i,n}_s dB_s - \int_t^T \int_{\mathbf{E}} l_t(e)\tilde{N}(dt, de) \tag{IV.7}$$

For $n \geq 1$ and $t \in [0, 1]$. For $n = 0$, we set $(X^{i,0}_s, Z^{i,0}_s, l^{i,0}_s) = (0, 0, 0)$.
Without loss of generality, we assume that f_1 satisfies Assumption 3.1, while f_2 is nondecreasing in x'.

Now we define

$$\tilde{f}_1^n(s,x,z,l) = f_1(s, F(s, X_s^{1,n-1}(\cdot)), x, z, l), \tag{IV.8}$$

$$\tilde{f}_2^n(s,x,z,l) = f_2(s, F(s, X_s^{2,n-1}(\cdot)), x, z, l) \tag{IV.9}$$

Then obviously we have $\tilde{f}_1^1 \leq \tilde{f}_2^1$ and $\tilde{f}_1^1$ satisfy the monotone assumption in Theorem 4.2 [11], (here I suggest put a subsection in our paper on the classic comparison results so that we can cite it directly as a lemma). Thus by the classic comparison theorem for BSDE with jumps (Theorem 4.2 [11]), we have

$$X_s^{1,1} \leq X_s^{2,1} a.s., \quad s \in [0,T]. \tag{IV.10}$$

Now since f_2 is nondecreasing in x', we have

$$\tilde{f}_1^2(s,x,z,l) = f_1(s, F(s, X_s^{1,1}(\cdot)), x, z, l) \tag{IV.11}$$

$$\leq f_2(s, F(s, X_s^{1,1}(\cdot)), x, z, l) \tag{IV.12}$$

$$\leq f_2(s, F(s, X_s^{2,1}(\cdot)), x, z, l) = \tilde{f}_2^2(s,x,z,l) \tag{IV.13}$$

where the last inequality follows from F is non decreasing. Using again the comparison results for classic BSDEs with jumps again, we get

$$X_s^{1,2} \leq X_s^{2,2}, \quad s \in [0,T].$$

By the same argument above, we can iteratively get that

$$X_s^{1,n} \leq X_s^{2,n}, \quad s \in [0,T]. \quad n \geq 1.$$

We now show that for $i = 1, 2$, $(X^{i,n}, Z^{i,n}, l^{i,n})_{n\geq 0}$ is Cauchy sequence in $\mathcal{S}^2 \times \mathbb{H}^2 \times^2_\nu$. Put $\bar{X}_t^{i,n} = X_t^{i,n} - X_t^{i,n-1}$; $\bar{Z}_t^{i,n} = Z_t^{i,n} - Z_t^{i,n-1}$; $\bar{l}_t^{i,n} = l_t^{i,n} - l_t^{i,n-1}$, applying Ito's formula to $e^{\beta s}|X_s^{i,n} - X_s^{i,n-1}|^2$, $n \geq 1$, we have analogously to the Proposition A.4 [11]

$$\begin{aligned} e^{\beta t}(\bar{X}_s^{i,n})^2 + \beta \int_t^T e^{\beta s}(\bar{X}_s^{i,n})^2 ds + \int_t^T e^{\beta s}(\bar{Z}_s^{i,n})^2 ds + \int_t^T e^{\beta s}\|\bar{l}_s^{i,n}\|_\nu^2 ds \\ = 2\int_t^T e^{\beta s}\bar{X}_s^{i,n}[f^i(s, F(s, X_s^{i,n-1}(\cdot)), X_s^{i,n}, Z_s^{i,n}, l_s^{i,n}) - f^i(s, F(s, X_s^{i,n-2}(\cdot)), Y_s^{i,n-1}, Z_s^{i,n-1}, l_s^{i,n-1})]ds \\ - 2\int_t^T e^{\beta s}\bar{X}_s^{i,n}\bar{Z}_s^{i,n} dW_s - \int_t^T e^{\beta s}\int_{\mathbb{R}^*}(2\bar{X}_{s^-}^{i,n}\bar{l}_s^{i,n}(u) + \bar{l}_s^{i,n}(u)^2) d\tilde{N}(ds, de) \end{aligned} \tag{IV.14}$$

Taking the conditional expectation given $\mathcal{F}_t$, we get

$$e^{\beta t}(\bar{X}_s^{i,n})^2+\mathbb{E}\left[\beta\int_t^T e^{\beta s}(\bar{X}_s^{i,n})^2ds+\int_t^T e^{\beta s}[(\bar{Z}_s^{i,n})^2+\|\bar{l}_s^{i,n}\|_\nu^2]ds \mid \mathcal{F}_t\right]$$
$$\leq 2\mathbb{E}\left[\int_t^T e^{\beta s}\bar{X}_s^{i,n}[f^i(s,F(s,X_s^{i,n-1}(\cdot)),X_s^{i,n},Z_s^{i,n},l_s^{i,n})-f^i(s,F(s,X_s^{i,n-2}(\cdot)),X_s^{i,n-1},Z_s^{i,n-1},l_s^{i,n-1})]ds \mid \mathcal{F}_t\right]. \tag{IV.15}$$

Moreover,

$$\begin{aligned}|f^i(s,F(s,X_s^{i,n-1}(\cdot)),X_s^{i,n},Z_s^{i,n},l_s^{i,n})-f^i(s,F(s,X_s^{i,n-2}(\cdot)),X_s^{i,n-1},Z_s^{i,n-1},l_s^{i,n-1})|\\ \leq C[|F(s,X_s^{i,n-1})-F(s,X_s^{i,n-2})|+|\bar{X}_s^{i,n}|+|\bar{Z}_s^{i,n}|+\|\bar{l}_s^{i,n}\|_\nu^2]\\ \leq C_1[\|\bar{X}_s^{i,n-1}\|_2+|\bar{X}_s^{i,n}|+|\bar{Z}_s^{i,n}|+\|\bar{l}_s^{i,n}\|_\nu^2]=C_1[(\mathbb{E}|\bar{X}_s^{i,n-1}|^2)^{\frac{1}{2}}+|\bar{X}_s^{i,n}|+|\bar{Z}_s^{i,n}|+\|\bar{l}_s^{i,n}\|_\nu^2]\end{aligned} \tag{IV.16}$$

Now, for all real numbers x', x, z, l and $\varepsilon>0$
$2x(Cx+Cz+Cl)\leq\frac{y^2}{\varepsilon^2}+\varepsilon^2(Cx+Cz+Cl)^2\leq\frac{x^2}{\varepsilon^2}+3\varepsilon^2(C^2x^2+C^2z^2+C^2l^2)$. and $\mathbb{E}[2\bar{X}_s^{i,n}(\mathbb{E}|\bar{X}_s^{i,n-1}|^2)^{\frac{1}{2}}]\leq\mathbb{E}[\frac{1}{\eta^2}|\bar{X}_s^{i,n}|^2+\eta^2\mathbb{E}|\bar{X}_s^{i,n-1}|^2]=\frac{1}{\eta^2}\mathbb{E}|\bar{X}_s^{i,n}|^2+\eta^2\mathbb{E}|\bar{X}_s^{i,n-1}|^2$
Thus we obtain that

$$e^{\beta t}(\bar{X}_s^{i,n})^2+\mathbb{E}\left[\beta\int_t^T e^{\beta s}(\bar{X}_s^{i,n})^2ds+\int_t^T e^{\beta s}[(\bar{Z}_s^{i,n})^2+\|\bar{l}_s^{i,n}\|_\nu^2]ds \mid \mathcal{F}_t\right]$$
$$\leq\mathbb{E}\left[\frac{C}{\eta^2}\int_t^T e^{\beta s}(\bar{X}_s^{i,n})^2ds+C\eta^2\int_t^T e^{\beta s}(\bar{X}_s^{i,n-1})^2ds+3C^2\epsilon^2\int_t^T e^{\beta s}[(\bar{Z}_s^{i,n})^2+\|\bar{l}_s^{i,n}\|_\nu^2]ds \mid \mathcal{F}_t\right] \tag{IV.17}$$

Choose $\eta=\epsilon$, $\epsilon^2=\frac{1}{3C^2+\frac{1}{2}C}$ and $\beta=\frac{C}{\epsilon^2}+\frac{1}{2}C\epsilon^2$.
Then we have (β,η,ϵ) satisfying $1-3C^2\epsilon^2\geq\frac{1}{2}C\eta^2$ and $\beta-\frac{C}{\eta^2}\geq\frac{1}{2}C\eta^2$, which gives contraction inequality:

$$\|\bar{X}^{i,n}\|_\beta^2+\|\bar{Z}^{i,n}\|_\beta^2+\|\bar{l}^{i,n}\|_{\nu,\beta}^2\leq\frac{1}{2}(\|\bar{X}^{i,n-1}\|_\beta^2+\|\bar{Z}^{i,n-1}\|_\beta^2+\|\bar{l}^{i,n-1}\|_{\nu,\beta}^2) \tag{IV.18}$$

If we denote the bound of $\|\bar{X}^{i,1}\|_\beta^2+\|\bar{Z}^{i,1}\|_\beta^2+\|\bar{l}^{i,1}\|_{\nu,\beta}^2$ by M, then by iteration we have,

$$\|\bar{X}^{i,k}\|_\beta^2+\|\bar{Z}^{i,k}\|_\beta^2+\|\bar{l}^{i,k}\|_{\nu,\beta}^2\leq\frac{M}{2^{k-1}}$$

which means that $(X^{i,n},Z^{i,n},l^{i,n})$ converges in $\mathcal{S}^2\times\mathbb{H}^2\times\mathbb{H}_\nu^2$ to some (X^i,Z^i,l^i). Now taking limit in (IV.7), we see that (X^i,Z^i,l^i) is the unique solution of (IV.5). And $X_t^1\leq X_t^2, t\in[0,T]a.s.$ follows directly from the fact that $X_t^{1,n}\leq X_t^{2,n}, t\in[0,T]a.s.$ □

Remark 1.6. *We can weaken the nondecreasing property of F: let $D_1(x)=\mathbb{P}(X_1\leq x)$ and $D_2(x)=\mathbb{P}(X_2\leq x)$. Then we call F non decreasing in x, if $D_1(x)\geq D_2(x)$ implies $F(t,X_1)\leq F(t,X_2)$. In particular, in the example of Remark 1.4, when $F(X)=\mathbb{E}(\phi(t,X))$,* *for $X \in L^2(\Omega)$, F is non decreasing if ϕ is C^1, non decreasing. This can be verified by direct computation : $F(t,X_1) - F(t,X_2) = \mathbb{E}(\phi(t,X_1)) - \mathbb{E}(\phi(t,X_2)) = \int \phi(t,x)d(D_1 - D_2)(x) = \int \frac{\partial\phi}{\partial x}(t,x)(D_2 - D_1)(x)dx = \int \frac{\partial\phi}{\partial x}(t,x)[D_2(x) - D_1(x)]dx$. We also notice that $X_1 \le X_2$ a.s. implies that $D_1(x) \ge D_2(x)$.*

Remark 1.7. *Symmetrically, if we assume one of the drivers is non-increasing in x'; and F is an non-increasing operator, the arguments in Theorem 1.2 still hold.*

Theorem 1.3 (Strict comparison for Mean-field BSDEs with jumps)**.** *Suppose the assumption of Theorem 1.2 holds. Moreover we assume one of the coefficient satisfy Assumption 3.1 with strict inequality $\theta_t^{x',x,z,l^1,l^2}(u) > -1$ $dt \otimes dP$- a.s. If $X^1_{t_0} = X^2_{t_0}$ a.s. for some $t_0 \in [0,T]$, then $X^1_. = X^2_.$ a.s. on $[t_0,T]$.*

Proof. Put $\bar{X}_s = X^1_s - X^2_s$; $\bar{Z}_s = Z^1_s - Z^2_s$; $\bar{l}_t(u) = l^1_s(u) - l^2_s(u)$; We then suppose that f_2 is nondecreasing in x'. We furthermore Assumption 3.1 with strict inequality holds for f_1. Then

$$-dX_s = h_s ds - \bar{Z}_s dW_s - \int_{\mathbf{E}} \bar{l}_s(e)\tilde{N}(ds,de). \quad \bar{X}_T = \xi_1 - \xi_2.$$

where $h_s := f_1(s, F(s,X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s(\cdot)) - f_2(s, F(s,X^2_s(\cdot)), X^2_s, Z^2_s, l^2_s(\cdot))$.
Let $\phi(s) := f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^2_s, l^2_s) - f_2(s, F(s,X^2_s(\cdot)), X^2_s, Z^2_s, l^2_s)$.

We have $h_s = \phi_s + f_1(s, F(s,X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s) - f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^2_s, l^2_s)$

We can write $f_1(s, F(s,X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s) - f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^2_s, l^2_s) =$
$f_1(s, F(s,X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s) - f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^1_s, l^1_s) + f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^1_s, l^1_s)$
$- f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^2_s, l^1_s) + f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^2_s, l^1_s) - f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^2_s, l^2_s)$.
Then from the Assumption 3.1 on f_1, there exists bounded processes δ and β on $\bar{\Omega} \times [0,T]$, such that

$$f_1(s, F(s,X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s) - f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^2_s, l^2_s) \ge \delta_s \bar{X}_s + \beta_s \bar{Z}_s + \langle \theta_s, \bar{l}_s \rangle_\nu$$

with

$$\delta_s := \frac{f_1(s, F(s,X^1_s(\cdot)), X^1_s, Z^1_s, l^1_s) - f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^1_s, l^1_s)}{\bar{X}_s}$$

$$\beta_s := \frac{f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^1_s, l^1_s) - f_1(s, F(s,X^1_s(\cdot)), X^2_s, Z^2_s, l^1_s)}{\bar{Z}_s}$$

and θ_s is as in Assumption 3.1.
Thus we have $h_s \ge \phi_s + \delta_s \bar{X}_s + \beta_s \bar{Z}_s + \langle \theta_s, \bar{l}_s \rangle_\nu$. For each $t \in [0,T]$, let $(\Gamma_{t,s})_{s\in[t,T]}$ be the unique solution of the forward SDE

$$d\Gamma_{t,s} = \Gamma_{t,s^-}\big[\delta_s ds + \beta_s dW_s + \int_{\mathbf{E}} \theta_s(e)\tilde{N}(dt,de)\big]; \quad \Gamma_{t,t} = 1.$$

By the comparison results with respect to a linear BSDE (see lemma 4.1 in [11]) we can derive that

$$\bar{X}_{t_0} \ge \mathbb{E}\big[\Gamma_{t_0,t}\bar{X}_t + \int_{t_0}^t \Gamma_{t_0,s}\phi(s)ds|\mathcal{F}_{t_0}\big], \quad t_0 \le t \le T.$$

Due to the Theorem 1.2, we have $\bar{X}_t = X^1_t - X^2_t \geq 0$ and the nondecreasing property of F, we can write

$$f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, l^2_s) \geq f_2(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, l^2_s) \geq f_2(s, F(s, X^2_s(\cdot)), X^2_s, Z^2_s, l^2_s)$$

by the assumption on f_1 and f_2. Then we can conclude the proof by pointing out the fact that

$$\phi(s) = f_1(s, F(s, X^1_s(\cdot)), X^2_s, Z^2_s, l^2_s) - f_2(s, F(s, X^2_s(\cdot)), X^2_s, Z^2_s, l^2_s) \geq 0$$

and that if $\theta_s(u) > -1 dP \otimes ds \otimes d\nu(u)$-a.s., then $\Gamma_{t,s} > 0$ a.s. from Corollary 3.5 in [11]. □

Remark 1.8. *In fact we can weaken the assumption by assuming the strict inequality* $\theta_t^{x',x,z,l^1,l^2}(u) > -1$ *holds only through the solutions ,that is* $\theta_t^{F(t,X^1_t),X^2_t,Z^2_t,l^1_t,l^2_t}(u) > -1\ dt \otimes dP$*-a.s. In the symmetric case when* f_2 *is Lipshitz and nondecreasing in* x'*, we get the results by assumming* $\theta_t^{F(t,X^2_t),X^2_t,Z^2_t,l^1_t,l^2_t}(u) > -1\ dt \otimes dP$*- a.s.*

2 Global dynamic risk measures

2.1 Definition and properties

Let $T > 0$ be a time horizon and f be a Lipschitz driver. For each $T' \in [0, T]$ and $\eta \in L^2(\mathcal{F}_{T'})$, set

$$\rho^{f,F}_t(\eta, T') = \rho_t(\eta, T') := -X_t(\eta, T'), \quad 0 \leq t \leq T', \tag{IV.19}$$

where $X_t(\eta, T')$ denotes the solution of mean-field the BSDE (IV.5) with driver f, mean-field operator F and terminal conditions (T', η). If T' represents a given maturity and η a financial position at time T', then $\rho_t(\eta, T')$ is interpreted as the risk of η at time t. The functional $\rho : (\eta, T') \mapsto \rho.(\eta, T')$ thus represents a *mean-field dynamic risk measure* induced by the mean-field BSDE with driver f and mean-field operator F. We now provide properties of these dynamic risk measures. Such as monotonicity, translation invariance, convexity are satisfied under appropriate hypotheses on the driver. Contrary to the Brownian case, the *monotonicity* property of ρ, that is, the non increasing property with respect to financial position, which is naturally required for risk measures, is not automatically satisfied. We thus assume from now on that the driver f satisfies the Assumption 3.1. Contrary to the standard non mean- field case, the zero-one law is not satisfied due to the mean-field operator F.

- *Continuity.* Let $T \in [0, T']$, let $\{\theta^\alpha, \alpha \in R\}$ be a family of stopping time in $\mathbb{T}_{0,T}$, converging a.s. to a stopping time $\theta \in \mathbb{T}_{0,T}$ as α tends to α_0. let $\{\xi^\alpha, \alpha \in R\}$ be a family of random variables such that $\mathbb{E}[\operatorname{esssup}_\alpha(\xi^\alpha)^2] < \infty$, and for each α, ξ^α is $\mathcal{F}_{\theta^\alpha}$ measurable. Suppose also that ξ^α converges a.s. to an $\mathcal{F}_{\theta^\alpha}$ measurable random variable ξ as α tends to α_0. Then for each $S \in \mathbb{T}_{0,T}$, the random variable $\rho_S(\xi^\alpha, \theta^\alpha) \to \rho_S(\xi, \theta)$ a.s. and the processes $\rho(\xi^\alpha, \theta^\alpha) \to \rho_S(\xi, \theta)$ in $S^{2,T}$ when $\alpha \to \alpha_0$. For the proof, see Appendix 4.4.

- *Monotonicity.* ρ is nonincreasing with respect to ξ. i.e. for each $T \in [0, T']$, and each $\xi^1, \xi^2 \in L^2(\mathcal{F}_T)$. if $\xi^1 \geq \xi^2$ a.s., then $\rho_t(\xi^1, T) \leq \rho_t(\xi^2, T)$, $0 \leq t \leq T$ a.s.
  Proof. This is the direct consequence of the comparison results of mean-field BSDEs (see Theorem 1.2) □

- *Consistency.* By the flow property, ρ is consistent: more precisely, Let $T \in [0, T']$ and let $S \in \mathbb{T}_{0,T}$ be a stopping time, then for each time t smaller than S, the risk-measure associated with position ξ and maturity T coincides with risk-measure associated with maturity S and position $-\rho_S(\xi, T) = X_S(\xi, T)$, that is

$$\forall t \leq S, \rho_t(\xi, T) = \rho_t(-\rho_S(\xi, T), S) \quad \text{a.s.}$$

The flow property is the consequence of the uniqueness result of the mean-field BSDEs

- *Translation invariance (cash additivity).* If f does not depend on x' and x, then the associated risk- measure satisfies the following *translation invariance* property:

$$\rho_t(\xi + \xi', T) = \rho_t(\xi, T) - \xi', \quad \text{for any } \xi \in L^2(\mathcal{F}_T) \text{ and } \xi' \in L^2(\mathcal{F}_T)$$

- *Cash sub-additivity.* Suppose f is non-decreasing in both x' and x, while F is non-decreasing operator. Then for any $m > 0$, we have $\rho_t(\xi + m, T) \leq \rho_t(\xi, T) + m$.
Proof. It is straightforward that $X_t + m$ satisfying:

$$-d(X_t + m) = f(t, F(t, X_t), X_t, Z_t, l_t)dt - Z_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt, de); X_T = \xi + m.$$

Thus, by the assumption, we have

$$-d(X_t + m) \leq f(t, F(t, X_t + m), X_t + m, Z_t, l_t)dt - Z_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt, de); X_T = \xi + m.$$

let $(\bar{X}, \bar{Z}, \bar{l})$ be a solution of the Mean-field BSDE

$$-d\bar{X}_t = f(t, F(t, \bar{X}_t), \bar{X}_t, \bar{Z}_t, \bar{l}_t)dt - \bar{Z}_t dW_t - \int_{\mathbf{E}} \bar{l}_t(e)\tilde{N}(dt, de); \bar{X}_T = \xi + m.$$

Then by the (extended) comparison results, we have $X_t + m \leq \bar{X}_t$, which gives $\rho_t(\xi + m, T) \leq \rho_t(\xi, T) + m$. □

- *Convexity.* If f is concave with respect to (x', x, z, l). We furthermore assume f nondecreasing in x' and F is nondreasing concave operator in x, then the dynamic risk-measure ρ is convex, that is for any $\lambda \in [0, 1]$, $T \in [0, T']$, $\xi^1, \xi^2 \in L^2(\mathcal{F}_t)$

$$\rho(\lambda\xi^1 + (1 - \lambda)\xi^2, T) \leq \lambda\rho(\xi^1, T) + (1 - \lambda)\rho(\xi^2, T).$$

Proof. For $i = 1, 2$, let (X^i, Z^i, l^i) be a solution of the mean-field BSDE (IV.5) associated to terminal time T, driver f, Mean-field operator F and terminal condition ξ^i. Set $\hat{\xi} := \lambda\xi^1 + (1 - \lambda)\xi^2$, $\hat{X} := \lambda X^1 + (1 - \lambda)X^2$, $\hat{Z} := \lambda Z^1 + (1 - \lambda)Z^2$, $\hat{l} := \lambda l^1 + (1 - \lambda)l^2$.

We have

$$-d\hat{X}_t = [\lambda f(t, F(t, X_t(\cdot)), X_t, Z_t, l_t(\cdot)) + (1-\lambda) f(t, F(t, X_t(\cdot)), X_t, Z_t, l_t(\cdot))]dt - \hat{Z}_t dW_t - \int_{\mathbf{E}} \hat{l}_t(e)\tilde{N}(dt, de);$$
$$\hat{X}_T = \hat{\xi}.$$

By the assumption on f, and F, we have

$$\begin{aligned} &\lambda f(t, F(t, X_t^1), X_t^1, Z_t^1, l_t^1) + (1-\lambda) f(t, F(t, X_t^2), X_t^2, Z_t^2, l_t^2) \\ &\le f(t, \lambda F(t, X_t^1) + (1-\lambda)F(t, X_t^2), \lambda X_t^1 + (1-\lambda)X_t^2, \lambda Z_t^1 + (1-\lambda)Z_t^2, \lambda l_t^1 + (1-\lambda) l_t^2) \\ &\le f(t, F(t, \lambda X_t^1 + (1-\lambda)X_t^2), \lambda X_t^1 + (1-\lambda)X_t^2, \lambda Z_t^1 + (1-\lambda)Z_t^2, \lambda l_t^1 + (1-\lambda) l_t^2) \\ &= f(t, F(t, \hat{X}_t), \hat{X}_t, \hat{Z}_t, \hat{l}_t) \end{aligned}$$

Thus

$$-d\hat{X}_t \le f(t, F(t, \hat{X}_t), \hat{X}_t, \hat{Z}_t, \hat{l}_t)dt - \hat{Z}_t dW_t - \int_{\mathbf{E}} \hat{l}_t(e)\tilde{N}(dt, de); \hat{X}_T = \hat{\xi}.$$

let $(\bar{X}, \bar{Z}, \bar{l})$ be a solution of the mean-field BSDE (IV.5) associated to terminal time T, driver f, mean-field operator F and terminal condition $\hat{\xi}$. i.e.

$$-d\bar{X}_t = f(t, F(t, \bar{X}_t), \bar{X}_t, \bar{Z}_t, \bar{l}_t)dt - \bar{Z}_t dW_t - \int_{\mathbf{E}} \bar{l}_t(e)\tilde{N}(dt, de); \bar{X}_T = \hat{\xi}.$$

by (extended) comparison results Th.1.2, we obtain $\hat{X}_t \le \bar{X}_t$ which gives the results. □

Suppose furthermore in Assumption 3.1 we have $\theta_t^{x',x,z,l^1,l^2}(u) > -1$.

- *No Arbitrage*. For each $T \in [0, T']$, and $\xi^1, \xi^2 \in L^2(\mathcal{F}_T)$, if $\xi^1 \ge \xi^2$ a.s. and if $\rho_t(\xi^1, T) = \rho_t(\xi^2, T)$ a.s. on an event $A \in \mathcal{F}_t$. then $\xi^1 = \xi^2$ a.s. on A.
  This is the directly consequence of strict comparison results (Theorem 1.3).

In the classical case without the mean field term and when the drift term is independent of the current position, the risk measure is cash additive. With the mean field, it is understood that cash is transferred via the mean field term. When f non-decreasing in the mean field term, then the mean field is acting as a stabilizer. This means that individually, a bank needs less capital to reach the same final position as without the stabilizing effect.

In systemic risk models where the mean field has a stabilizing effect [1, 4–6], the function f captures interbank lending: banks with high liquidity position lend to banks with low liquidity position, where high and low is given with respect to the average liquidity in the system. Here, we do not consider formally a model of interbank lending, but our risk processes behave similarly. In addition, when f is convex we give a dual representation of the risk measures.

Interpretation as systemic risk measure .

We may regard (IV.5) as a limit as $n \mapsto \infty$ of the following system:

$$
\begin{aligned}
& -dX_t^{i,N} = f(t,\omega,\frac{1}{N}\sum_{j=1}^N X_t^{j,N}(\cdot), X_t^{i,N}, Z_t^{i,N}, l_t^{i,N}(\cdot))dt - Z_t^{i,N}dW_t - \int_{\mathbf{E}} l_t^{i,N}(e)\tilde{N}(dt,de); \\
& X_T^{i,N} = \xi.
\end{aligned}
\tag{IV.20}
$$

Here $-X_t^{i,N}$ represents the liquidity or capital reserve of bank i. In a system framework, its evolution depends on the capital of the other banks through the mean capital reserve. This stylized dependence structure captures well a situation where a joint liquidity fund is used to stabilize banks. Such is the case of a central clearinghouse for example. We can think of this setup as a regulator choosing a level $-\xi$ for the acceptable capital or liquidity at a horizon T (ξ is then the financial position at time t). Before time t and in order to become acceptable at the time horizon, the capital (liquidity) of the individual bank may have inflows/outflows from the common pool via the mean field term.

2.2 Dual representation of convex global risk measures

We now provide a representation for global dynamic risk measures induced by concave mean-field BSDEs in terms of the value of a stochastic control problem. In this case, the risk measure is convex. This dual representation is given via a control problem over a set of probability measures which are absolutely continuous with respect to P.

Let f be a Lipschitz driver and F be an Lipschitz operator. For each (ω, t), we denote by f^* the Fenchel-Legendre transform of f, defined for each $(\beta, q, \alpha_1, \alpha_2) \in \mathbb{R}^3 \times L_\nu^2$ and F^* the Fenchel-Legendre transform of F, defined for each $\delta \in L_{\mathbf{P}}^2$, that is,

$$
f^*(\omega,t,q,\beta,\alpha_1,\alpha_2) = \sup_{(x',x,z,l)\in\mathbb{R}^3\times L_\nu^2} [f(\omega,t,x',x,z,l) - qx' - \beta x - \alpha_1 z - \langle \alpha_2, l\rangle_\nu]
$$

$$
F^*(t,\delta) = \sup_{X\in L_{\mathbf{P}}^2} \Big[F(t,X) - \langle X,\delta\rangle_{L_{\mathbf{P}}^2}\Big]
$$

For each predictable processes $\alpha_t = (\alpha_t^1, \alpha_t^2(\cdot))$, let $\mathcal{Q}^\alpha$ be the probability absolutely continuous with respect to $\mathcal{P}$ which admits Γ_T^α as density with respect to P on $\mathcal{F}_T$, where Z^α is the solution of

$$
d\Gamma_t^\alpha = \Gamma_{t-}^\alpha\big(\alpha_t^1 dW_t + \int_{\mathbb{R}^*} \alpha_t^2(u) d\tilde{N}(dt,du)\big); \quad \Gamma_0^\alpha = 1. \tag{IV.21}
$$

Now we introduce $\mathcal{A}_T$ be the set of predictable processes $\alpha_s = (\alpha_s^1, \alpha_s^2)$ such that

- $\int_0^T (\alpha_s^1)^2 ds + \int_0^T \|\alpha_s^2\|_\nu^2 ds$ is bounded
- $\alpha_s^2(u) > -1 \quad \nu(du) - a.s.$

We have from Proposition 3.1 and 3.2 [11] that for all $\alpha_t \in \mathcal{A}_T$, $\Gamma_t^\alpha > 0, 0 \le t \le T$ $a.s.$ and $(\Gamma_t^\alpha)_{0\le t\le T} \in S^{2,T}$.
And let $\bar{\mathcal{A}}_T$ be the set of processes $(\gamma_t, \beta_t, q_t, \alpha_t^1, \alpha_t^2)$ where $(\beta_t, q_t, \alpha_t^1, \alpha_t^2)$ are predictable and γ_t is progressively measurable, such that

- $f^*(\omega, t, q_t, \beta_t, \alpha_t^1, \alpha_t^2)$ belongs to $\mathbb{H}^2$
- $\alpha_t = (\alpha_t^1, \alpha_t^2(\cdot))$ belongs to $\mathcal{A}_T$.
- $0 \le q_t \le C \quad dP$ a.s.
- The processes $\Gamma_t^\alpha e^{-\int_0^t \gamma_s ds}$ belongs to $\mathbb{H}^2$

In the sequel, we assume that f **is nondecreasing with respect to** x' and satisfies Assumption 3.1 with strict inequality $\theta_t(u) > -1$ $dt \otimes dP$- a.s.

2.2.1 Technical lemmas

We begin by the following technical lemma on part of the bounds of the control set which appear in the definition of the Fenchel transform of f.

Lemma 2.1. *For each (s,ω), $D_s(\omega)$ is the non empty set of $(\bar{q}, \bar{\beta}, \bar{\alpha}_1, \bar{\alpha}_2) \in \mathbb{R}^3 \times L_\nu^2$ such that $f^*(\omega, s, \bar{q}, \bar{\beta}, \bar{\alpha}_1, \bar{\alpha}_2) < +\infty$. Then $D_s(\omega)$ is included in the set Usatisfying the following properties:*

- *$q \ge 0$ and is bounded by C.*
- *β and α_1 are bounded by C.*
- *$\alpha_2(u) > -1 \quad and \quad |\alpha_2(u)| \le C \quad \nu(du) - a.s.$*

where C is the Lipschitz constant of f.

Proof. Let us suppose that $q < 0$, we will show that this assumption leads to contradiction. By the definition of f^* we have

$$f^*(t, q, \beta, \alpha_1, \alpha_2) \ge f(t, x', 0, 0, 0) - x'q,$$

which holds for each x'. This holds in particular for $x_n := n$ $n \in \mathbb{N}$. We thus gets

$$f^*(t, q, \beta, \alpha_1, \alpha_2) \ge f(t, n, 0, 0, 0) - nq \ge f(t, 0, 0, 0, 0) - nq$$

where the last inequality follows by the non-decreasingness of the map f with respect to x'. By letting $n \to +\infty$ in the above inequality, we get $\lim_{n \to +\infty} f(t, 0, 0, 0, 0) - qn = +\infty$, since $q < 0$. This implies that $f^*(t, q, \beta, \alpha_1, \alpha_2) < +\infty$ which provides the expected contradiction, We thus have proved that $q \ge 0$. The fact that q, β and α are included in the bounded domain $[-C, C]$ is due to the uniform Lipschitz property of f. Finally, for the properties of (α_1, α_2), we apply the similar proof as in Lemma 5.4 [11]. Suppose by contradiction that

$$\nu(\{u \in \mathbb{R}^*, \alpha_2(u) \le -1\}) > 0.$$

Since f satisfies Assumption 3.1, the following inequality holds for each $l \in L_\nu^2$.

$$f(\omega, t, 0, l) \ge f(\omega, t, 0, 0) + \langle \theta_t^{0,l,0}(\omega), l \rangle_\nu$$

. Again by the definition of f^* we have

$$f^*(\omega, t, \alpha_1, \alpha_2) \ge f(\omega, t, 0, l) - \langle \alpha_2, l \rangle_\nu \ge f(\omega, t, 0, 0) + \langle \theta_t^{0,l,0}(\omega) - \alpha_2, l \rangle_\nu$$

. This holds in particular for $l := n\mathbf{1}_{\{\alpha_2 \leq -1\}}$ where $n \in \mathbb{N}$. We thus derive

$$f^*(\omega, t, \alpha_1, \alpha_2) \geq f(\omega, t, 0, 0) + n \int_{\{\alpha_2 \leq -1\}} \big(\theta_t^{0,l,0}(\omega, u) - \alpha_2(u)\big)\nu(du)$$

Since by assumption, $\theta_t^{0,l,0}(\omega, u) > -1$, thus $\theta_t^{0,l,0}(\omega, u) - \alpha_2(u) > 0$ on $\{\alpha_2 \leq -1\}$. By letting n tend to $+\infty$ in this inequality, we get $f^*(\omega, t, \alpha_1, \alpha_2) = +\infty$, which provides the expected contradiction. We thus have proven that $\alpha_2 > -1$ ν-a.s. By similar arguments, one can show that α_1 is bounded by C and $|\alpha_2(u)| \leq C \quad \nu(du) - a.s.$,which ends the proof. □

The following technical lemma is used in the proof of the dual representation Theorem 2.1. It provides part of the bounds of the control set which appear in the definition of the Fenchel transform of F.

Lemma 2.2. *If the operator F is nondecreasing as in Theorem 1.2. Then for each $t \in [0, T]$, the non empty set of $\{\delta \in L^2_{\mathbf{P}} | F^*(t, \delta) < +\infty\}$is in included in the set satisfying the following properties:*

- *$\delta \geq 0$ dP a.s. and $\|\delta\|_{L^2_{\mathbf{P}}} \leq C$. where C is the Lipschitz constant of F.*

Proof. Suppose $\delta \geq 0$ $d\mathbf{P}$ a.s.is not true. We denote $A = \{\omega \in \Omega | \delta(\omega) < 0\}$ Then $P(A) > 0$. By the definition of F^*, we have for each $X \in L^2_{\mathbf{P}}$

$$F^*(t, \delta) \geq F(t, X) - \langle X, \delta \rangle_{L^2_{\mathbf{P}}} = F(t, X) - \mathbb{E}^{\mathbf{P}}[X\delta]$$

This holds in particular for $X_n(\omega) := -n\delta\mathbf{1}_A(\omega)$ where $n \in \mathbb{N}$. This gives $X_n \geq 0$ $d\mathbb{P}$ a.s. and thus by the nondecreasing properties of F, we obtain

$$F^*(t, \delta) \geq F(t, X^n) - \mathbb{E}^{\mathbf{P}}[X^n\delta] \geq F(t, 0) - \mathbb{E}^{\mathbf{P}}[X^n\delta] = F(t, 0) + n \int_A |\delta(\omega)|^2 d\mathbf{P}(\omega)$$

By letting $n \to +\infty$ in the above inequality, we get $F^*(\delta) = +\infty$, which gives the contradiction to the assumption. Thus $\mathbf{P}(A) = 0$ which implies $\delta \geq 0$ $d\mathbf{P}$ a.s. The boundedness of δ is a direct results of the Lipschitz property of F. □

2.2.2 Dual representation theorem

We now give the main result of this section, the dual representation theorem of the mean risk measure. Note that at time zero, this is the risk measure itself.

For $(\gamma, q, \beta, \alpha^1, \alpha^2) \in \bar{\mathcal{A}}_T$, we denote $D^{\beta,\gamma}_{t,s} := \exp(-\int_t^s (\beta_u + \gamma_u \mathbf{1}_{\mathbb{E}[q_u]>0})du), 0 \leq t \leq s \leq T$, which can be interpreted as a discount factor. We recall that Γ^α_s follows the dynamics defined in (IV.21).

The following lemma will be used for the existence of the optimal control that appears in the dual representation theorem. In the classical setting without the mean field, three components $\tilde{q}_s, \bar{\beta}_s, \bar{\alpha}_s$ of the optimal control are shown to exist in previous literature. The challenge we now solve is to show that given these three optimal components, we can construct the fourth component which is associated to the mean field operator.

Lemma 2.3. *Given the predictable processes processes $(X_s, \tilde{q}_s, \bar{\beta}_s, \bar{\alpha}_s)_{s\geq t}$, there exist a progressively measurable processes $(\bar{\gamma}_s)_{s\geq t}$ such that $(\Gamma^{\bar{\alpha}_s} e^{-\int_t^s \gamma_u du})_{s\geq t}$ belongs to $\mathbb{H}^2$ and satisfying the following equation:*

$$F(s, X_s) - \frac{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[X_s D_{t,s}^{\bar{\beta},\bar{\gamma}} \bar{\gamma}_s]}{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D_{t,s}^{\bar{\beta},\bar{\gamma}} \tilde{q}_s]} = F^*(s, \frac{\Gamma_s^{\bar{\alpha}} D_{t,s}^{\bar{\beta},\bar{\gamma}} \bar{\gamma}_s}{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D_{t,s}^{\bar{\beta},\bar{\gamma}} \tilde{q}_s]}) \tag{IV.22}$$

Proof. Since F is concave and Lipschitz on $L^2(\Omega, \mathbf{P}, \mathcal{F})$, the conjugacy relation of (F, F^*) gives that for each s, there exists $Y_s \in L^2(\Omega, \mathbf{P}, \mathcal{F}_s)$ such that

$$F(s, X_s) - \mathbb{E}^{\mathbf{P}}[X_s Y_s] = F^*(s, Y_s) \tag{IV.23}$$

and since F is nondecreasing, by lemma 2.2 we have $Y_s \geq 0$ $d\mathbf{P}$ a.s. and $\|Y_s\|_{L_{\mathbf{P}}} \leq C$. Now suppose $(V_s)_{s\geq t} \in \mathcal{S}$ be the solution of (IV.36) with $U_s = e^{\int_t^s \bar{\beta}_u du} Y_s$, $h_s = e^{-\int_t^s \bar{\beta}_u du} \tilde{q}_s$ and $V_t = 1$. We apply Ito's formula to $V_s(\Gamma_s^{\bar{\alpha}})^{-1}$, we have

$$d(V(\Gamma^{\bar{\alpha}})^{-1})_s = (\Gamma_s^{\bar{\alpha}})^{-1} e^{\int_0^s \bar{\beta}_u du} Y_s \mathbb{E}[V_s e^{-\int_0^s \bar{\beta}_u du} \tilde{q}_s] ds \tag{IV.24}$$

Thus $V^1 := V(\Gamma^{\bar{\alpha}})^{-1}$ satisfies the random differential equation

$$d(V^1)_s = (\Gamma_s^{\bar{\alpha}})^{-1} e^{\int_0^s \bar{\beta}_u du} Y_s \mathbb{E}[\Gamma_s^{\bar{\alpha}} V_s^1 e^{-\int_0^s \bar{\beta}_u du} \tilde{q}_s] ds \tag{IV.25}$$

Since $\Gamma_s^{\bar{\alpha}},\ \tilde{q}_s > 0$, $Y_s \geq 0$ $d\mathbf{P}$ a.s., if $V_t^1 := V_t = 1 > 0$ a.s. we have $V_s^1 > 0$ a.s. Thus for each $(s, \omega) \in [t, T] \times \Omega$, we can choose $\gamma_s(\omega) = -\frac{d}{ds}(\log V_s^1)(\omega)$ which is well defined dues to (IV.25). This gives the process $\bar{\gamma}$ satisfying that $e^{-\int_t^s \bar{\gamma}_u du} = V_s^1$ and from (IV.25) we obtain that

$$\bar{\gamma}_s e^{-\int_t^s \bar{\gamma}_u du} ds = d(e^{-\int_t^s \bar{\gamma}_u du})_s = (\Gamma_s^{\bar{\alpha}})^{-1} e^{\int_t^s \bar{\beta}_u du} Y_s \mathbb{E}[\Gamma_s^{\bar{\alpha}} e^{-\int_t^s \bar{\gamma}_u du} e^{-\int_t^s \bar{\beta}_u du} \tilde{q}_s] ds \tag{IV.26}$$

which implies $(\bar{\gamma}_s)_{s\geq t}$ satisfies

$$\frac{\Gamma_s^{\bar{\alpha}} D_{t,s}^{\bar{\beta},\bar{\gamma}} \bar{\gamma}_s}{\mathbb{E}[\Gamma_s^{\bar{\alpha}} D_{t,s}^{\bar{\beta},\bar{\gamma}} \tilde{q}_s]} = Y_s \ a.s.$$

And $\Gamma_s^{\bar{\alpha}} e^{-\int_t^s \bar{\gamma}_u du} = V_s$ belongs to $\mathbb{H}^2$.
Now we show that $(\bar{\gamma}_s)_{s\geq t}$ is progressively measurable. Since the Hilbert space $L^2_{\mathbf{P}}(\mathcal{F}_T)$ is separable, hence by the measurable selection theorem, the processes

$$Y : [t, T] \mapsto L^2_{\mathbf{P}}(\mathcal{F}_T), s \mapsto Y_s(\cdot)$$

are measurable with respect to $\mathcal{B}([t, T])$ and $\mathcal{B}(L^2_{\mathbf{P}}(\mathcal{F}_T))$. Furthermore, for each Y belongs to $L^2_{\mathbf{P}}(\mathcal{F}_s)$ and feasible, by the property satisfied by the F, we have for each $X \in L^2_{\mathbf{P}}(\mathcal{F}_T)$, and $t \in [0, T]$.

$$F(t, X) - \mathbb{E}^{\mathbf{P}}[XY] \leq F(t, \mathbb{E}[X|\mathcal{F}_s]) - \mathbb{E}^{\mathbf{P}}[\mathbb{E}[X|\mathcal{F}_s]Y]$$

This gives

$$F^*(t, Y) = \sup_{X \in L^2_{\mathbf{P}}(\mathcal{F}_T)} [F(t, X) - \langle X, Y \rangle_{L^2_{\mathbf{P}}}] = \sup_{X \in L^2_{\mathbf{P}}(\mathcal{F}_s)} [F(t, X) - \langle X, Y \rangle_{L^2_{\mathbf{P}}}]$$

Thus we could restrict the operator F on the subspace $L^2_{\mathbf{P}}(\mathcal{F}_s)$. This implies we could choose $Y_s \in L^2_{\mathbf{P}}(\mathcal{F}_s)$ in (IV.23) and for each $u \in [0,T]$, $Y : [t,u] \mapsto L^2_{\mathbf{P}}(\mathcal{F}_u), s \mapsto Y_s(\cdot)$ are measurable with respect to $\mathcal{B}([t,u])$ and $\mathcal{B}(L^2_{\mathbf{P}}(\mathcal{F}_u))$. Now for each $u \in [t,T]$, since $L^2_{\mathbf{P}}(\mathcal{F}_u)$ is a separable Hilbert space, there exists a countable orthonormal basis $(e^i_u)_{i\in\mathbb{N}}$ of $L^2_{\mathbf{P}}(\mathcal{F}_u)$. For each $i \in \mathbb{N}$, define $\lambda^i_u = \langle Y_u, e^i\rangle_{\mathbf{P}}$. Since the map $\langle ., e^i_u\rangle_{\mathbf{P}}$ is continuous on $L^2_{\mathbf{P}}(\mathcal{F}_u)$, $\lambda^i : [t,u] \mapsto \mathbb{R}, s \mapsto \lambda^i_s$ is $\mathcal{B}([t,u])$ measurable. As $Y_s(\omega) = \sum_i \lambda^i_s e_i(\omega)$, it follows that $Y : [t,u]\times\Omega \mapsto \mathbb{R}, (s,\omega) \mapsto Y_s(\omega)$ is $\mathcal{B}([t,u])\otimes\mathcal{F}_u$ measurable. This holds for each $u \in [t,T]$, thus $(Y_s)_{s\geq t}$ is progressive measurable. And this gives the progressive measurability of $(\bar{\gamma}_s)_{s\geq t}$. □

We are now ready to give the main result. The risk measure can be interpreted as the expectation (under a worst-case discount factor and the worst case probability measure) of the final position ξ plus a penalty function. The lemmas of the previous subsection ensure that the supremum is finite as the relevant domain is bounded. Lemma 2.3 gives the existence of the optimal control.

Theorem 2.1. *Suppose that the Hilbert space $L^2_{\mathbf{P}}$ and L^2_ν are separable. Let f be a Lipschitz driver with Lipschitz constant C satisfying Assumption 3.1 and Suppose also f is concave with respect to (x',x,z,l) and non-decreasing in x'. F is a Lipschitz concave operator in x and satisfies the following property: for each $s,t \in [0,T]$ and $X \in L^2_{\mathbf{P}}$, $F(t,\mathbb{E}[X|\mathcal{F}_s]) \geq F(t,X)$. Let $(\gamma,q,\beta,\alpha^1,\alpha^2) \in \bar{\mathcal{A}}_T$, we denote $D^{\beta,\gamma}_{t,s} := exp(-\int_t^s(\beta_u+\gamma_u\mathbf{1}_{\mathbb{E}[q_u]>0})du), 0\leq t\leq s\leq T$, and recall that Γ^α_s follows the dynamics defined in (IV.21).Then the mean of the convex risk-measure $\mathbb{E}\rho(.,T)$ has the following representation : for each $\xi \in L^2_{\mathbf{P}}(\mathcal{F}_T)$,*

$$\mathbb{E}\rho_t(\xi,T) = \sup_{(\gamma,\beta,q,\alpha)\in\bar{\mathcal{A}}_T}\left[\mathbb{E}^{\mathcal{Q}^\alpha}D^{\beta,\gamma}_{t,T}(-\xi) - \zeta(\gamma,\beta,q,\alpha,T)\right] \tag{IV.27}$$

where the function ζ, called penalty function, is defined, for each T and $(\gamma,q,\beta,\alpha^1,\alpha^2) \in \bar{\mathcal{A}}_T$ by

$$\zeta(\gamma,\beta,q,\alpha,T) := \int_t^T\left(\mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s}f^*(s,q_s,\beta_s,\alpha_s)] + \mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s}q_s]F^*(t,\frac{\Gamma^\alpha_s D^{\beta,\gamma}_{t,s}\gamma_s}{\mathbb{E}^{\mathcal{Q}^\alpha}[D^{\beta,\gamma}_{t,s}q_s]})\mathbf{1}_{\mathbb{E}^{\mathcal{Q}^\alpha}[q_s]>0}\right)ds$$

Moreover, for each $\xi \in L^2_{\mathbf{P}}(\mathcal{F}_T)$, there exists $(\bar{\gamma}_t,\bar{q}_t,\bar{\beta}_t,\bar{\alpha}^1_t,\bar{\alpha}^2_t) \in \bar{\mathcal{A}}_T$ achieving the supreme in (IV.27).

Proof. For each processes $(\gamma_s,q_s,\beta_s,\alpha^1_s,\alpha^2_s) \in \bar{\mathcal{A}}_T$, we apply Ito's formula to $D^{\beta,\gamma}_{t,s}X_s$ between t and T, where (X,Z,l) is the solution of Mean-field BSDE (IV.5). We obtain

$$X_t = D^{\beta,\gamma}_{t,T}\xi + \int_t^T D^{\beta,\gamma}_{t,s}[-\beta_sX_s - \gamma_s\mathbf{1}_{\mathbb{E}[q_s]>0}X_s - \alpha^1_sZ_s - \langle\alpha^2_s,l_s\rangle_\nu + f(s,F(s,X_s(\cdot)),X_s,Z_s,l_s)]ds - \int_t^T dM^{\mathcal{Q}^\alpha}_s \tag{IV.28}$$

where $dM_s^{\mathcal{Q}^\alpha} = D_{t,s}^{\beta,\gamma} Z_s dW_s^{\mathcal{Q}^\alpha} + \int_{\mathbf{E}} D_{t,s}^{\beta,\gamma} l_s(e) d\tilde{N}^{\mathcal{Q}^\alpha}(dt, de)$. For each $s \in [t,T]$, we have

$$\begin{aligned}
&-\beta_s X_s - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s - \alpha_s^1 Z_s - \langle \alpha_s^2, l_s \rangle_\nu + f(s, F(s, X_s(\cdot)), X_s, Z_s, l_s) \\
&= -\beta_s X_s - q_s F(s, X_s) - \alpha_s^1 Z_s - \langle \alpha_s^2, l_s \rangle_\nu + f(s, F(s, X_s(\cdot)), X_s, Z_s, l_s) + (q_s F(s, X_s) - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s)
\end{aligned} \tag{IV.29}$$

Since $q_s \geq 0$ $d\mathbf{P}$ a.s., we notice that

$$q_s F(s, X_s) - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s = (q_s F(s, X_s) - \gamma_s X_s) \mathbf{1}_{\mathbb{E}[q_s]>0}$$

By taking expectation at time $t = 0$ on both sides, we can obtain that

$$\begin{aligned}
\mathbb{E}[X_t] = \mathbb{E}^{\mathcal{Q}^\alpha} &\left[D_{t,T}^{\beta,\gamma} \xi + \int_t^T D_{t,s}^{\beta,\gamma} [-\beta_s X_s - q_s F(X_s) - \alpha_s^1 Z_s - \langle \alpha_s^2, l_s \rangle_\nu + f(s, F(s, X_s(\cdot)), X_s, Z_s, l_s)] \right] ds \\
&+ \int_t^T \mathbb{E}^{\mathcal{Q}^\alpha}[D_{t,s}^{\beta,\gamma} q_s] \left[F(s, X_s) - \frac{\mathbb{E}^{\mathcal{Q}^\alpha}[X_s D_{t,s}^{\beta,\gamma} \gamma_s]}{\mathbb{E}^{\mathcal{Q}^\alpha}[D_{t,s}^{\beta,\gamma} q_s]} \right] \mathbf{1}_{\mathbb{E}[q_s]>0} ds
\end{aligned} \tag{IV.30}$$

holds for all processes $(\gamma_s, \beta_s, q_s, \alpha_s^1, \alpha_s^2) \in \bar{\mathcal{A}}_T$. Since $\mathcal{Q}^{\bar{\alpha}}$ and $\mathbf{P}$ are equivalent measures and $q_s \geq 0$ $d\mathbf{P}$ a.s., we have $\mathbf{1}_{\mathbb{E}[q_s]>0} = \mathbf{1}_{\mathbb{E}^{\mathcal{Q}^\alpha}[q_s]>0}$

By the definition of Fenchel-Legendre transform, we have $f(s, F(s, X_s), X_s, Z_s, l_s) - q_s F(s, X_s) - \beta_s X_s - \alpha_s^1 Z_s - \langle \alpha_s^2, l_s \rangle_\nu \leq f^*(s, q_s, \beta_s, \alpha_s^1, \alpha_s^2)$ a.s. and $F(s, X_s) - \frac{\mathbb{E}^{\mathcal{Q}^\alpha}[X_s D_{t,s}^{\beta,\gamma} \gamma_s]}{\mathbb{E}^{\mathcal{Q}^\alpha}[D_{t,s}^{\beta,\gamma} q_s]} = F(s, X_s) - \frac{\mathbb{E}[\Gamma_s^\alpha X_s D_{t,s}^{\beta,\gamma} \gamma_s]}{\mathbb{E}[\Gamma_s^\alpha D_{t,s}^{\beta,\gamma} q_s]} \leq F^*(s, \frac{\Gamma_s^\alpha D_{t,s}^{\beta,\gamma} \gamma_s}{\mathbb{E}^{\mathcal{Q}^\alpha}[D_{t,s}^{\beta,\gamma} q_s]})$

Since by assumption $D_{t,s}^{\beta,\gamma} \geq 0$ and $q_s \geq 0$ $d\mathcal{Q}^\alpha$ a.s., we can obtain

$$\begin{aligned}
\mathbb{E} X_t \leq \inf_{(\beta,\gamma,q,\alpha) \in \mathcal{A}_T} \mathbb{E}^{\mathcal{Q}^\alpha} &\left[D_{t,T}^{\beta,\gamma} \xi + \int_t^T D_{t,s}^{\beta,\gamma} f^*(s, q_s, \beta_s, \alpha_s^1, \alpha_s^2) \right] ds \\
+ \int_t^T \mathbb{E}^{\mathcal{Q}^\alpha}[D_{t,s}^{\beta,\gamma} q_s] &\left[F^*(s, \frac{\Gamma_s^\alpha D_{t,s}^{\beta,\gamma} \gamma_s}{\mathbb{E}^{\mathcal{Q}^\alpha}[D_{t,s}^{\beta,\gamma} q_s]}) \right] \mathbf{1}_{\mathbb{E}^{\mathcal{Q}^\alpha}[q_s]>0} ds
\end{aligned} \tag{IV.31}$$

Recall that for any $(\omega, s) \in \Omega \times [0,T]$, f is Lipschitz, concave in (x', x, z, l), the following conjugacy relation of (f, f^*) holds. Let U be the set introduced in Lemma 2.1, we have $D_s(\omega) \subset U$. Using the same arguments as in Lemma 5.5 in [11], we have

$$\begin{aligned}
f(\omega, s, x', x, z, l) &= \inf_{(q,\beta,\alpha^1,\alpha^2) \in \bar{U}} \{ f^*(\omega, s, q, \beta, \alpha^1, \alpha^2) + q x' + \beta x + \alpha^1 z + \langle \alpha^2, l \rangle_\nu \} \\
&= f^*(\omega, s, \bar{q}, \bar{\beta}, \bar{\alpha}^1, \bar{\alpha}^2) + \bar{q} x' + \bar{\beta} x + \bar{\alpha}^1 z + \langle \bar{\alpha}^2, l \rangle_\nu
\end{aligned} \tag{IV.32}$$

where $\bar{U}$ is the closure of set U, that is the set in which α_2 satisfies $\alpha_2(u) \geq -1$ instead of the strict inequality.

Now since $\bar{U}$ is strongly closed and convex, we obtain there exists $(\bar{q}, \bar{\beta}, \bar{\alpha}^1, \bar{\alpha}^2) \in \bar{U}$ that satisfy (IV.32). However Lemma 2.1 gives in fact that $(\bar{q}, \bar{\beta}, \bar{\alpha}^1, \bar{\alpha}^2) \in D_s(\omega)$.

Since, by assumption, $\mathbb{R}^3 \times L_\nu^2$ is separable, we can apply the measurable selection theorem

(Appendix of Ch.III [2]) as in the lemma 5.5 of [11] to assert the existence of a predictable processes $(\bar{q}_s, \bar{\beta}_s, \bar{\alpha}^1_s, \bar{\alpha}^2_s)_{s\geq t}$

$$f(s, F(s, X_s(\cdot)), X_s, Z_s, l_s) = \bar{\beta}_s X_s + \bar{q}_s F(X_s) + \bar{\alpha}^1_s Z_s + \langle \bar{\alpha}^2_s, l_s\rangle_\nu + f^*(s, \bar{\beta}_s, \bar{q}_s, \bar{\alpha}^1_s, \bar{\alpha}^2_s) \quad a.s. \tag{IV.33}$$

Similarly, since F is Lipschitz and concave, the conjugacy relation also holds for (F, F^*). Given the predictable processes $(X_s, \bar{q}_s, \bar{\beta}_s, \bar{\alpha}_s)_{s\geq t} \in \mathcal{S} \times \bar{\mathcal{A}}_T$, we now introduce $\tilde{q}_s = \bar{q}_s \mathbf{1}_{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[\bar{q}_s]>0} + C\mathbf{1}_{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[\bar{q}_s]=0}$ with C the Lipschitz constant of f. We can show (lemma 2.3) there exist a predictable processes $(\bar{\gamma}_s)_{s\geq t} \in \bar{\mathcal{A}}_T$ such that

$$F(s, X_s) - \frac{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[X_s D^{\bar{\beta},\bar{\gamma}}_{t,s}\bar{\gamma}_s]}{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma}}_{t,s}\tilde{q}_s]} = F^*(s, \frac{\Gamma^{\bar{\alpha}}_s D^{\bar{\beta},\bar{\gamma}}_{t,s}\bar{\gamma}_s}{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma}}_{t,s}\tilde{q}_s]}) \tag{IV.34}$$

And since $\tilde{q}_s = \bar{q}_s$ for any s such that $\mathbb{E}[\bar{q}_s] > 0$, thus we obtain

$$\mathbb{E}X_t = \mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}\left[D^{\bar{\beta},\bar{\gamma}}_{t,T}\xi + \int_t^T D^{\bar{\beta},\bar{\gamma}}_{t,s} f^*(s, \bar{\beta}_s, q_s, \bar{\alpha}^1_s, \bar{\alpha}^2_s)\right] ds + \int_t^T \mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma}}_{t,s}\bar{q}_s]\left[F^*(s, \frac{\Gamma^{\bar{\alpha}}_s D^{\bar{\beta},\bar{\gamma}}_{t,s}\bar{\gamma}_s}{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma}}_{t,s}\bar{q}_s]})\right]\mathbf{1}_{\mathbb{E}^{\mathcal{Q}^{\bar{\alpha}}}[\bar{q}_s]>0} ds \tag{IV.35}$$

Together with (IV.31), we obtain (IV.27).
Finally, (IV.33) implies that the processes $f^*(\omega, t, \bar{q}_t, \bar{\beta}_t, \bar{\alpha}^1_t, \bar{\alpha}^2_t)$ belongs to $\mathbb{H}^2_T$ since by assumption $(X, Z, l(.)) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ and $(\bar{q}_t, \bar{\beta}_t, \bar{\alpha}^1_t, \bar{\alpha}^2_t)$ are bounded. □

Remark 2.4. *We give an example of the F such that the assumptions in Theorem 2.1 hold. Define $F(t, X) := \mathbb{E}[\phi(t, X)]$ for $X \in L^2(\Omega, \boldsymbol{P}, \mathcal{F}_T)$, where $\phi : [0, T] \times \mathbb{R} \mapsto \mathbb{R}, (t, x) \mapsto \phi(t, x)$ is a concave Lipschitz function such that $\phi(t, X) \in L^2_{\mathbb{P}}$. Then by conditional Jensen's inequality*

$$\mathbb{E}\left[\phi(t, \mathbb{E}[X|\mathcal{F}_s])\right] \geq \mathbb{E}\left[(\mathbb{E}[\phi(t, X)|\mathcal{F}_s])\right] = \mathbb{E}[\phi(t, X)]$$

An example is take $\phi(t, x) = -(x - k_s)^-$ where $(k_s)_{s\geq 0}$ is a deterministic function.

Lemma 2.5. *Suppose $(\alpha^1_s, \alpha^2_s(\cdot))_{s\geq t}$ belongs to $\mathcal{A}_T$, $(U_s)_{s\geq t}$ are bounded in $L^2_{\boldsymbol{P}}$ and $(h_s)_{s\geq t}$ are bounded almost surely. Then the following SDE admits a solution $(V_s)_{s\geq t} \in \mathcal{S}$.*

$$\begin{aligned} dV_s = V_s[\alpha^1_s dW_s + \int_{\mathbb{R}^*} \alpha^2_s(u) d\tilde{N}(ds, de)] + U_s\mathbb{E}[V_s h_s]ds,\ t \leq s \leq T \\ V_t = 1 \end{aligned} \tag{IV.36}$$

Proof. We define inductively the sequence (V^n) of the processes by setting $V^0 \equiv V_0$ and for $n \geq 1$

$$V^n_u = V_t + \int_t^u V^{n-1}_s dM_s + \int_t^u U_s\mathbb{E}[V^{n-1}_s h_s]ds$$

where $dM_s = \alpha^1_s dW_s + \int_{\mathbb{R}^*} \alpha^2_s(e) d\tilde{N}(ds, de)$ Because any two real numbers h and k satisfy

$(h+k)^2 \leq 2(h^2+k^2)$, we have

$$\mathbb{E}\left[\sup_{t\leq s\leq u}|V_s^{n+1}-V_s^n|^2\right] \leq 2\mathbb{E}\left[\sup_{t\leq s\leq u}\left(\int_0^s (V_s^n - V_s^{n-1})dM_s\right)^2 + \sup_{t\leq s\leq u}\left(\int_0^s U_s\mathbb{E}[V_s^n - V_s^{n-1}]ds\right)^2\right] \tag{IV.37}$$

And by Doob and Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq s\leq u}|V_s^{n+1}-V_s^n|^2\right] &\leq 8\mathbb{E}\left[\left(\int_0^u (V_s^n - V_s^{n-1})dM_s\right)^2\right] + 2t\mathbb{E}\left[\int_0^u |U_s\mathbb{E}[V_s^n - V_s^{n-1}]|^2 ds\right]\\ &\leq 8\mathbb{E}\left[\int_0^u (V_s^n - V_s^{n-1})^2 d[M,M]_s\right] + 2t\left[\int_0^u \mathbb{E}|U_s|^2|\mathbb{E}[V_s^n - V_s^{n-1}]|^2 ds\right]\end{aligned} \tag{IV.38}$$

Since $d[M,M]_s = (\alpha_s^1)^2 ds + \int_{\mathbb{R}^*}(\alpha_s^2(e))^2 d\tilde{N}(ds,de) + \int_{\mathbb{R}^*}(\alpha_s^2(e))^2\nu(de)ds$ and by assumption, $(\alpha_s^1, \alpha_s^2(\cdot))_{s\geq t}$ belongs to $\mathcal{A}_T$, $(U_s)_{s\geq 0}$ are bounded in $L^2(\mathbf{P})$, we obtain there exists a constant K such that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq s\leq u}|V_s^{n+1}-V_s^n|^2\right] &\leq 2K(4+t)\mathbb{E}\left[\int_0^u |V_s^n - V_s^{n-1}|^2 ds\right]\\ &\leq 2K(4+t)\mathbb{E}\left[\int_0^u \sup_{t\leq s\leq u}|V_s^n - V_s^{n-1}|^2 ds\right]\end{aligned} \tag{IV.39}$$

Now for fixed T, we set $C = 2K(4+T)$ and let $D := \mathbb{E}\left[\sup_{t\leq s\leq T}|V_s^1 - V_s^0|^2\right]$. It then follows from the above computation that for each $t \leq u \leq T$ and n,

$$\mathbb{E}\left[\sup_{t\leq s\leq u}|V_s^n - V_s^{n-1}|^2\right] \leq \frac{DC^nT^n}{n!}$$

consequently

$$\sum_{n=1}^{\infty}\left\|\sup_{t\leq s\leq u}|V_s^n - V_s^{n-1}|\right\|_2 < \infty$$

Thus it gives the series $\sum_{n=1}^{\infty}\sup_{t\leq s\leq u}|V_s^n - V_s^{n-1}|$ converges a.s. and as a results, X^n converges a.s. uniformly on every bounded interval to a continuous process V which is a solution to (IV.36) $\square$

3 Optimization principle for Mean-field BSDEs

Let ξ in $\mathcal{S}^2$ and let $(f, f^\alpha; \alpha \in \mathcal{A})$ be a family of Lipschitz drivers satisfying Assumption 3.1 and F is Lipschitz operator. In Assumption 3.1, the coefficient associated with f^α (resp. f), is denoted by $\theta_t^{\alpha,x',x,z,l}$ (resp. $\theta_t^{x',x,z,l}$). And from this section on, we furthermore assume that :

- f is nondecreasing in x'

- F is nondecreasing.

We denote by (X, Z, l) the solution of the mean-field BSDE associated to obstacle (ξ_t) and driver f, and by $(X^\alpha, Z^\alpha, l^\alpha)$ the solution of the mean-field BSDE associated with obstacle (ξ_t) and driver f^α.

Proposition 3.1 (Optimization principle for mean-field BSDEs)**.** *Suppose that*

(i) *For each* $\alpha \in \mathcal{A}$, $f(t, x', x, z, l) \leq f^\alpha(t, x', x, z, l)$, *for all* $(x', x, z, l) \in \mathbb{R}^3 \times \mathcal{L}^2_\nu$; $dt \otimes dP - a.s.$

(ii) *There exists* $\bar{\alpha} \in \mathcal{A}$ *such that*

$$f(t, F(t, X_t(\cdot)), X_t, Z_t, l_t) = f^{\bar{\alpha}}(t, F(t, X_t(\cdot)), X_t, Z_t, l_t), \quad 0 \leq t \leq T, \quad dt \otimes dP - \text{a.s.} \tag{IV.40}$$

Then, for each $S \in \mathbb{T}_0$,

$$Y_S = \operatorname*{ess\,inf}_\alpha Y_S^\alpha = Y_S^{\bar{\alpha}} \quad \text{a.s.} \tag{IV.41}$$

Proof. For each α, since Condition 1. is satisfied, f^α satisfies Assumption 3.1, and since f is nondecreasing in x' , F is nondecreasing, the comparison theorem for mean-field BSDEs yields (see Theorem 1.2) that $Y \leq Y^\alpha$. It follows that for each $S \in \mathbb{T}_0$,

$$Y_S \leq \operatorname*{ess\,inf}_\alpha Y_S^\alpha \quad \text{a.s.}$$

Now, by Condition 2. , Y is a solution of the mean-field BSDE associated with $f^{\bar{\alpha}}$. By uniqueness of the solution of this mean-field BSDE, we have $Y = Y^{\bar{\alpha}}$, which leads to equality (IV.41).
□

CHAPTER V

Optimal stopping for Mean-Field BSDEs with jumps

1 Reflected Mean-Field BSDEs with jumps

1.1 Notation and definitions

Definition 1.1. *A process $(Y,Z,k(.),A)$ is said to be a solution of the reflected BSDE associated with driver f and obstacle $\xi_.$ if*

$$\begin{aligned}
&(Y,Z,k(.),A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2\\
&-dY_t = f(t,Y_t,Z_t,k_t(\cdot))dt + dA_t - Z_t dW_t - \int_U k_t(u)\tilde{N}(dt,du); \quad Y_T=\xi_T, \qquad \text{(V.1)}\\
&Y_t \geq \xi_t,\ 0\leq t\leq T \ a.s.,\\
&A \text{ is a nondecreasing RCLL predictable process with } A_0=0 \text{ and such that}\\
&\int_0^T (Y_t-\xi_t)dA_t^c = 0 \ a.s. \ and \ \Delta A_t^d = -\Delta Y_t \mathbf{1}_{\{Y_{t^-}=\xi_{t^-}\}} \quad \text{a.s.}
\end{aligned}$$

Here A^c denotes the continuous part of A and A^d its discontinuous part.

Definition 1.2. *A process $(Y,Z,k(.),A)$ is said to be a solution of the Mean-field reflected BSDE with jump associated with driver f, operator F and obstacle $\xi_.$ if*

$$\begin{aligned}
&(Y,Z,k(.),A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2\\
&-dY_t = f(t,\omega,F(t,X_t(\cdot)),X_t,Z_t,k_t(\cdot))dt + dA_t - Z_t dW_t - \int_U k_t(u)\tilde{N}(dt,du); \quad Y_T=\xi_T, \qquad \text{(V.2)}\\
&Y_t \geq \xi_t,\ 0\leq t\leq T \ a.s.,\\
&A \text{ is a nondecreasing RCLL continous process with } A_0=0 \text{ and such that}\\
&\int_0^T (Y_t-\xi_t)dA_t^c = 0 \ a.s. \ and \ \Delta A_t^d = -\Delta Y_t \mathbf{1}_{\{Y_{t^-}=\xi_{t^-}\}} \quad \text{a.s.}
\end{aligned}$$

For $\beta>0$ and $\phi\in\mathbb{H}^{2,T}$, we introduce the norm $\|\phi\|^2_{\beta,T} := E[\int_0^T e^{\beta s}\phi_s^2 ds]$ and for $l\in\mathbb{H}^{2,T}_\nu$, we set $\|l\|^2_{\nu,\beta,T} := E[\int_0^T e^{\beta s}\|l_s\|^2_\nu\, ds]$.

Theorem 1.1. *([12] Theorem 2.6) Let $\xi_.$ be a RCLL adapted process in $\mathcal{S}^2$ and let f be a Lipschitz driver. The RBSDE (V.1) admits a unique solution $(Y,Z,k(.),A)\in\mathcal{S}^2\times\mathbb{H}^2\times\mathbb{H}^2_\nu\times\mathcal{S}^2$. If (ξ_t) is left-upper semicontinuous (l.u.s.c.) over stopping times, then (A_t) is continuous.*

We now show an existence and uniqueness result for mean-field reflected BSDEs with jumps, in the general case of RCLL obstacle.

Theorem 1.2. *(Existence and Uniqueness for Mean-field reflected BSDEs)*
Let $\xi_.$ be a RCLL adapted process in $\mathcal{S}^2$ and let f and F be Lipschitz driver and Lipshcitz Meanfield operator respectively. The Mean-field RBSDE (V.2) admits a unique solution $(Y, Z, k(.), A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$. If (ξ_t) is left-upper semicontinuous (l.u.s.c.) over stopping times, then (A_t) is continuous.

Proof. Denote by $\mathbb{H}^2_\beta$ the space $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ equipped with the norm $\|Y, Z, k(\cdot)\|^2_\beta := \|Y\|^2_\beta + \|Z\|^2_\beta + \|k\|^2_{\nu,\beta}$. We define a mapping Φ from $\mathbb{H}^2_\beta$ into itself as follows. Given $(U, V, l) \in \mathbb{H}^2_\beta$, let $(Y, Z, k) = \Phi(U, V, l)$ be the the solution of the standard RBSDE associated with driver $f^1(s) := f(s, F(s, U_s(\cdot)), U_s, V_s, l_s)$. Let A be the associated nondecreasing process. The mapping Φ is well defined by Theorem 1.1 .

Now we prove that the mapping Φ is a contraction from $\mathbb{H}^2_\beta$ into $\mathbb{H}^2_\beta$. Let $(\hat{U}, \hat{V}, \hat{l})$ be another element of $\mathbb{H}^2_\beta$ and let $(\hat{Y}, \hat{Z}, \hat{k}) := \Phi(\hat{U}, \hat{V}, \hat{l})$, that is, the solution of the RBSDE associated with driver process $f(s, F(s, \hat{U}_s(\cdot)), \hat{U}_s, \hat{V}_s, \hat{l}_s)$.

Set $\bar{U} = U - \hat{U}$, $\bar{V} = V - \hat{V}$, $\bar{l} = l - \hat{l}$, $\bar{Y} = Y - \hat{Y}$, $\bar{Z} = Z - \hat{Z}$, $\bar{k} = k - \hat{k}$.

Let $\Delta f_. := f(\cdot, F(s, U_s(\cdot)), U_s, V_s, l_s) - f(\cdot, F(s, \hat{U}_s(\cdot)), \hat{U}_s, \hat{V}_s, \hat{l}_s)$. From Lipschitz continuity of f, we have $\mathbb{E}|\Delta f|^2 = \mathbb{E}|f(\cdot, F(s, U_s(\cdot)), U_s, V_s, l_s) - f(\cdot, F(s, \hat{U}_s(\cdot)), \hat{U}_s, \hat{V}_s, \hat{l}_s)|^2 \leq 4C^2\mathbb{E}[|\overline{U}|^2 + |\overline{V}|^2 + \|\bar{l}\|^2_\nu]$. Here we have used the fact that $|F(s, U_s(\cdot)) - F(s, \hat{U}'_s(\cdot))|^2 \leq C\|U - \hat{U}\|^2_2 = \mathbb{E}|\overline{U}|^2$ Now recall that $\|\Delta f\|^2_\nu = [\int_0^T e^{\beta s}\mathbb{E}|\Delta f|^2\, ds]$. Using estimates (V.31) and (V.32) (see also (A.58) and (A.59) in [12]) with $\eta \leq \frac{1}{2C^2}$ and Lipschitz constant equal to 0 (since the driver f^1 does not depend on the solution), we get

$$\|\bar{Y}\|^2_\beta + \|\bar{Z}\|^2_\beta + \|\bar{k}\|^2_{\nu,\beta} \leq \eta(T+2)\|\Delta f\|^2_\beta \leq \eta(T+2)4C^2(\|\bar{U}\|^2_\beta + \|\bar{V}\|^2_\beta + \|\bar{l}\|^2_{\nu,\beta}),$$

Choosing $\eta = \frac{1}{(T+2)8C^2}$, we deduce $\|(\overline{Y}, \overline{Z}, \bar{k})\|^2_\beta \leq \frac{1}{2}\|(\overline{U}, \overline{V}, \bar{l})\|^2_\beta$. Hence, Φ is a contraction and thus admits a unique fixed point (Y, Z, k) in $\mathbb{H}^2_\beta$, which is the solution of RBSDE (V.1). For the second assertion, assuming that ξ is l.u.s.c. over stopping times, we now show that for any predictable stopping time $\tau \in \mathbb{T}_0$, $\Delta A_\tau = 0$ a.s. Since $\Delta A^d_t = -\Delta Y_t\, \mathbf{1}_{\{Y_{t^-} = \xi_{t^-}\}}$, we obtain

$$\Delta A^d_\tau = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(Y_{\tau^-} - Y_\tau)^+ = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(\xi_{\tau^-} - Y_\tau)^+ \leq \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(\xi_\tau - Y_\tau)^+$$

The last inequality follows from the inequality $\xi_{\tau^-} \leq \xi_\tau$ a.s. Since $\xi \leq Y$, we derive that $\Delta A^d_\tau \leq 0$ a.s. Thus due to the nondecreasing property of A, we have $\Delta A^d_\tau = 0$ for all predictable stopping time and the continuity of A follows directly. □

In this section, we show that similar comparison and strict comparison results to those given in Section 1.2 hold for mean field reflected BSDEs. Namely, given two derivers $f^1$and f^2 (resp. obstacle processes ξ^1 and ξ^2) satisfying a monotonicity condition, the associated solutions satisfy a monotonicity condition as well.

1.2 Comparison theorems for Mean-Field RBSDEs with jumps

Theorem 1.3 (Comparison)**.** *Let ξ^1, ξ^2 be two RCLL obstacle processes in $\mathcal{S}^2$. Let $f^1$and f^2 be Lipschitz drivers satisfying Assumption 3.1 and one of them is nondecreasing in x'. Let also F be a nondecreasing operator. Suppose that*

- $\xi_t^2 \leq \xi_t^1$, $0 \leq t \leq T$ *a.s.*
- $f^1(\omega, t, x', x, z, l(\cdot)) \geq f^2(\omega, t, x', x, z, l(\cdot))$ $dP \otimes dt$,*a.s.* *for all* $(x', x, z, l(\cdot)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times L^2_\nu$

Let (Y^i, Z^i, k^i, A^i) *be the solution of the RBSDE associated with* (ξ^i, f^i) , $i = 1, 2$ *and mean-field operator* F. *Then,*

$$Y_t^2 \leq Y_t^1, \ \forall t \in [0, T] \ \text{a.s.}$$

Proof. We give a simple proof based on the characterization of solutions of mean-field RBSDEs (Theorem 2.1) and on the comparison theorem for non reflected mean-field BSDEs. Let $t \in [0, T]$. For each $\tau \in T_t$, let us denote by $X^i(\xi^i_\tau, \tau)$ the unique solution of the mean-field BSDE associated with (τ, ξ^i_τ, f^i) for $i = 1, 2$. By the comparison theorem for mean-field BSDEs, for each τ in $\mathbb{T}_t$, the inequality

$$X_t^2(\xi^2_\tau, \tau) \leq X_t^1(\xi^1_\tau, \tau) \ a.s.$$

holds. By taking the essential supremum over τ and using Theorem 2.1, we get

$$Y_t^2 = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}_t} X_t^2(\xi^2_\tau, \tau) \leq \operatorname*{ess\,sup}_{\tau \in \mathbb{T}_t} X_t^1(\xi^1_\tau, \tau) = Y_t^1 \ a.s. \quad \square$$

We now provide a strict comparison theorem. The first assertion addresses the particular case when the obstacle is l.u.s.c. along stopping times and the second one deals with the general case.

Theorem 1.4 (Strict comparison result for mean-field Reflected BSDE with jumps)**.** *Suppose that the assumptions of the comparison theorem (Theorem 1.3) hold and that the driver* f^1 *satisfies Assumption 3.1 with*

$$\theta_t^{x', x, z, l^1, l^2}(u) > -1 \quad dt \otimes dP - \text{a.s.} \tag{V.3}$$

Let S *in* $\mathbb{T}_0$ *and suppose that* $Y^1_S = Y^2_S$ *a.s.*

(i) *Suppose that* ξ^1 *and* ξ^2 *are l.u.s.c. along stopping times.* *Let* $\tau_i^* = \tau_{i,S}^* := \inf\{s \geq S;\, Y_s^i = \xi_s^i\}$, $i = 1, 2$. *Then,*

$$Y_t^1 = Y_t^2, \ \ S \leq t \leq \tau_1^* \wedge \tau_2^* \ \ a.s, \quad and$$

$$f^1(t, F(Y_t^1(\cdot)), Y_t^2, Z_t^2, k_t^2) \ = \ f^2(t, F(Y_t^2(\cdot)), Y_t^2, Z_t^2, k_t^2), \ \ S \leq t \leq \tau_1^* \wedge \tau_2^*, \ dP \otimes dt - a.s. \tag{V.4}$$

Moreover if $\xi^1 = \xi^2$ *a.s., then* $\tau_1^* = \tau_2^*$ *a.s. and* $Y^1_{\tau_1^*} = Y^2_{\tau_1^*} = \xi^1_{\tau_1^*}$ *a.s*

(ii) *Consider the general case when* ξ^1 *and* ξ^2 *are not supposed to be l.u.s.c. along stopping times. For* $\varepsilon > 0$, *define*

$$\tau_i^\varepsilon := \inf\{t \geq S, \ Y_t^i \leq \xi_t^i + \varepsilon\} \quad and \quad \tilde{\tau}_i := \lim_{\varepsilon \downarrow 0} \uparrow \tau_i^\varepsilon \ \ i = 1, 2.$$

Then, $Y_t^1 = Y_t^2$, $S \leq t < \tilde{\tau}_1 \wedge \tilde{\tau}_2$ *a.s. Moreover,*

$$f^1(t, F(Y_t^1(\cdot)), Y_t^2, Z_t^2, k_t^2) \ = \ f^2(t, F(Y_t^2(\cdot)), Y_t^2, Z_t^2, k_t^2) \ \ S \leq t \leq \tilde{\tau}_1 \wedge \tilde{\tau}_2, \ dP \otimes dt - a.s.$$

and if $\xi^1 = \xi^2$ a.s., then for each $\varepsilon > 0$, $\tau_1^\varepsilon = \tau_2^\varepsilon$ a.s. and $\tilde{\tau}_1 = \tilde{\tau}_2$.

Proof. 1. Let $i \in \{1, 2\}$. By the existence theorem (see Theorem 2.2), τ_i^* is optimal for Problem (V.6) with $f = f^i$, $\xi = \xi^i$, that is

$$Y_S^i = \text{ess}\sup_{\tau \in \mathbb{T}_S} X_S^i(\xi_\tau^i, \tau) = X_S^i(\xi_{\tau_i^*}^i, \tau_i^*) \quad \text{a.s}$$

where $X^i(\xi_{\tau_i^*}^i, \tau_i^*)$ denotes the solution of the mean-field BSDE associated with terminal time τ_i^*, terminal condition $\xi_{\tau_i^*}^i$ and driver f^i. Hence

$$Y_t^1 = X_t^1(Y_{\tau_1^* \wedge \tau_2^*}^1, \tau_1^* \wedge \tau_2^*), \text{ and } Y_t^2 = X_t^2(Y_{\tau_1^* \wedge \tau_2^*}^2, \tau_1^* \wedge \tau_2^*), \;\; S \leq t \leq \tau_1^* \wedge \tau_2^* \;\text{ a.s.}$$

Since $f^1 \geq f^2$ and $\xi^1 \geq \xi^2$, the comparison theorem for mean-field RBSDEs (Theorem 1.3) yields that $Y_{\tau_1^* \wedge \tau_2^*}^1 \geq Y_{\tau_1^* \wedge \tau_2^*}^2$ a.s. By hypothesis, $Y_S^1 = Y_S^2$. Now, Assumption (VII.13) allows us to apply the strict comparison theorem for non reflected mean-field BSDEs with jumps (Theorem 1.3) for terminal time $\tau_1^* \wedge \tau_2^*$. Hence, we get $Y_t^1 = Y_t^2$, $S \leq t \leq \tau_1^* \wedge \tau_2^*$ a.s., and equality (V.4), which provides the desired result.

Suppose now that $\xi^1 = \xi^2 = \xi$ a.s. Then, using $Y^2 \leq Y^1$, we get $\tau_2^* \leq \tau_1^*$ a.s. Since we have already shown that $Y_{\tau_2^*}^1 = Y_{\tau_2^*}^2$ a.s., and since $Y_{\tau_2^*}^2 = \xi_{\tau_2^*}$ a.s., we get $Y_{\tau_2^*}^1 = \xi_{\tau_2^*}$ and $\tau_1^* \leq \tau_2^*$ a.s. It follows that $\tau_1^* = \tau_2^*$ a.s.

2. Let $\varepsilon > 0$. By a property of τ_1^ε (see (V.11)), we have

$$Y_t^1 = X_t^1(Y_{\tau_1^\varepsilon}^1, \tau_1^\varepsilon), \;\; S \leq t \leq \tau_1^\varepsilon \;\text{ a.s.}$$

Similarly, $Y_t^2 = X_t^2(Y_{\tau_2^\varepsilon}^2, \tau_2^\varepsilon), S \leq t \leq \tau_2^\varepsilon$ a.s. By the same arguments as above with τ_1^* and τ_2^* replaced by τ_1^ε and τ_2^ε respectively, we derive the desired result.

Suppose now that $\xi^1 = \xi^2 = \xi$ a.s. Since $Y^2 \leq Y^1$, we have $\tau_2^\varepsilon \leq \tau_1^\varepsilon$ a.s. Moreover by the definition of ε-optimal stopping time and the strict comparison theorem for non-reflected mean-field BSDEs, we have $\xi_{\tau_2^\varepsilon} + \varepsilon \geq Y_{\tau_2^\varepsilon}^2 = Y_{\tau_2^\varepsilon}^1$ a.s. Consequently, $\tau_2^\varepsilon \geq \tau_1^\varepsilon$ a.s. and hence $\tau_2^\varepsilon = \tau_1^\varepsilon$ a.s. By letting ε tend to 0, we get $\tilde{\tau}_1 = \tilde{\tau}_2$ a.s. □

2 Optimal stopping for global dynamic risk measures

We start this section by describing the optimal stopping problem for global dynamic risk measures. For each stopping time $S \in \mathbb{T}_0$, we introduce the value function of the following optimization problem

$$Y_S = -\text{ess}\inf_{\tau \in \mathbb{T}_S} \rho_S(\xi_\tau, \tau) \quad \text{a.s.} \tag{V.5}$$

The aim of this section is to provide a representation for Y_S in terms of reflected mean-field BSDEs and to give the characterization of the optimal stopping strategy.

We now give the characterization result of the value function as the solution of a reflected mean-field BSDE.

Theorem 2.1 (Characterization of the value function)**.** *Let $T > 0$ be the terminal time. Let $(\xi_t, 0 \leq t \leq T)$ be an RCLL process in $\mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption*

3.1 and F is nondecreasing operator. We furthermore assume that f is nondecreasing in x', Suppose now that $(Y, Z, k(\cdot), A)$ is the solution of the mean-field reflected BSDE (V.2). Then

- *For each stopping time $S \in \mathbb{T}_0$, we have*

$$Y_S = \operatorname{ess\,sup}_{\tau \in \mathbb{T}_S} X_S(\xi_\tau, \tau) \quad \text{a.s.} \tag{V.6}$$

  where for $\tau \in \mathbb{T}_S$, $X.(\xi_\tau, \tau)$ is the solution of the Mean-field BSDE (IV.5) associated with terminal time τ, terminal condition ξ_τ, and driver f.

- *For each $S \in \mathbb{T}_0$ and each $\varepsilon > 0$, let τ_S^ε be the stopping time defined by*

$$\tau_S^\varepsilon = \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\}. \tag{V.7}$$

  We have

$$Y_S \ \leq \ X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) + K\varepsilon \quad \text{a.s.}, \tag{V.8}$$

  *where $K = K(T, C)$ is a constant which only depends on T and the Lipschitz constant C of f. In other words, τ_S^ε is a $(K\varepsilon)$-*optimal stopping time *for* (V.6).

Proof. For the reader's convenience, we give the proof below. Let $\tau \in \mathbb{T}_S$. Firstly we note that the process $(Y_s, Z_s, k_s; 0 \leq s \leq \tau)$ is the solution of the mean-field reflected BSDE (V.2) associated with terminal time τ, terminal condition Y_τ, and (generalized) driver

$$f(s, y', y, z, k)ds + dA_s.$$

We have $f(s, y', y, z, k)ds + dA_s \geq f(s, y', y, z, k)ds$ a.s. and $Y_\tau \geq \xi_\tau$ a.s.

Since f satisfies Assumption 3.1 and is nondecreasing in y', the comparison theorem 1.2 for mean-field BSDEs can be applied and gives $Y_s \geq X_s(\xi_\tau, \tau)$, $0 \leq s \leq \tau$ a.s. for each $\tau \in \mathbb{T}_S$. In particular $Y_S \geq X_S(\xi_\tau, \tau)$. By taking the supremum over $\tau \in \mathbb{T}_S$, we derive that

$$Y_S \geq \operatorname{ess\,sup}_{\tau \in \mathbb{T}_S} X_S(\xi_\tau, \tau) \quad \text{a.s.} \tag{V.9}$$

It remains to show the converse inequality. For each $S \in \mathbb{T}_0$ and for each $\varepsilon > 0$, let τ_S^ε be the stopping time defined by

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\}. \tag{V.10}$$

Firstly, by the definition of τ_S^ε and the right-continuity of (ξ_t) and (Y_t), we have

$$Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon \quad \text{a.s.}$$

Next we show that the process $(Y_t, S \leq t \leq \tau_S^\varepsilon)$ is the solution of the mean-field BSDE associated with terminal time τ_S^ε, terminal condition $Y_{\tau_S^\varepsilon}$, driver f and Mean-field operator F, that is

$$Y_t = X_t(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \quad S \leq t \leq \tau_S^\varepsilon \ \text{ a.s.} \tag{V.11}$$

We first note that $\tau_S^\varepsilon \in \mathbb{T}_S$. Fix $\varepsilon > 0$. By definition of τ_S^ε, for a.e. ω, if $t \in [S(\omega), \tau_S^\varepsilon(\omega)[$, then $Y_t(\omega) > \xi_t(\omega) + \varepsilon$ and hence $Y_t(\omega) > \xi_t(\omega)$. It follows that for a.e. ω, the function $t \mapsto A_t^c(\omega)$ is constant on $[S(\omega), \tau_S^\varepsilon(\omega)]$ and $t \mapsto A_t^d(\omega)$ is constant on $[S(\omega), \tau_S^\varepsilon(\omega)[$. Also,

$Y_{(\tau_S^\varepsilon)^-} \geq \xi_{(\tau_S^\varepsilon)^-} + \varepsilon$ a.s. Since $\varepsilon > 0$, it follows that $Y_{(\tau_S^\varepsilon)^-} > \xi_{(\tau_S^\varepsilon)^-}$ a.s., which implies that $\Delta A^d_{\tau_S^\varepsilon} = 0$ a.s. Hence, the process $(Y_t, S \leq t \leq \tau_S^\varepsilon)$ is a solution of the mean-field BSDE associated with terminal time τ_S^ε, terminal condition $Y_{\tau_S^\varepsilon}$ and driver f. By uniqueness of the solution of Lipschitz mean-field BSDEs, we get (V.11).
Finally we can prove inequality (V.8).

By (V.9) and by the comparison theorem for mean-field BSDEs, we derive that for each $\varepsilon > 0$,

$$Y_S = X_S(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \leq X_S(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) \quad \text{a.s.} \tag{V.12}$$

Now, by estimates on mean-field BSDEs (see Lemma 4.2 in Appendix), we have

$$|X_S(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) - X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon)|^2 \leq e^{\beta(T-S)}\varepsilon^2 \quad \text{a.s.}$$

where $\beta := 3C^2 + 4C$. This with (V.12) leads to inequality (V.8). Hence, for each $\varepsilon > 0$,

$$Y_S \;\leq\; X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) + K\varepsilon \;\leq\; \operatorname*{ess\,sup}_{\tau \in \mathbb{T}_S} X_S(\xi_\tau, \tau) + K\varepsilon \quad \text{a.s.} \tag{V.13}$$

where $K := e^{\frac{\beta T}{2}}$. It follows that $Y_S \;\leq\; \operatorname{ess\,sup}_{\tau \in \mathbb{T}_S} X_S(\xi_\tau, \tau)$ a.s. By (V.9), this inequality is an equality. Moreover, the $K\varepsilon$-optimality property of τ_S^ε follows from (V.13). □

The comparison results allow us to characterization the optimal stopping strategy and derive optimization principles for the case of mean-field BSDEs. We note that in a wide variety of papers staring with the seminal paper [3], the key steps for these results are the same given the comparison results. For completeness, we state these results with the addition of the mean field operator as well as proofs, for which we follow the proofs of [12].

Proposition 2.1 (Characterization of the optimal stopping strategy). *Let $(\xi_t, 0 \leq t \leq T)$ be a RCLL process in $\mathcal{S}^2$. Let f be a Lipschitz driver satisfying Assumption 3.1 and is nondecreasing in x', while F is the non-decreasing mean field operator. Let $S \in \mathbb{T}_0$ and let $\hat{\tau} \in \mathbb{T}_S$. Suppose that in Assumption 3.1, we have*

$$\theta_t^{F(t,Y_t),Y_t,Z_t,k_t,l_t^{\hat{\tau}}} > -1, \quad dt \otimes dP - \text{a.s.} \tag{V.14}$$

where $(X^{\hat{\tau}}, Z^{\hat{\tau}}, l^{\hat{\tau}}) = (X(\xi_{\hat{\tau}}, \hat{\tau}), Z(\xi_{\hat{\tau}}, \hat{\tau}), l(\xi_{\hat{\tau}}, \hat{\tau}))$ is the solution of the Mean-field BSDE associated with terminal conditions $(\hat{\tau}, \xi_{\hat{\tau}})$.
The stopping time $\hat{\tau}$ is S-optimal, i.e.

$$Y_S = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}_S} X_S(\xi_\tau, \tau) = X_S(\xi_{\hat{\tau}}, \hat{\tau}) \quad a.s. \tag{V.15}$$

if and only if

$$Y_s = X_s(\xi_{\hat{\tau}}, \hat{\tau}), \quad S \leq s \leq \hat{\tau} \quad \text{a.s.} \tag{V.16}$$

that is if and only if $(Y_s, S \leq s \leq \hat{\tau})$ is the solution of the non reflected Mean-field BSDE associated with terminal time $\hat{\tau}$ and terminal condition $\xi_{\hat{\tau}}$.

Proof. It is clear that (V.16) $\Rightarrow$ (V.15). (V.14). On the other hand, if we suppose (V.15) holds, then the process $(Y_s, Z_s, k_s; 0 \leq s \leq \hat{\tau})$ is the solution of the Mean-field BSDE associated with terminal conditions $\hat{\tau}$, $Y_{\hat{\tau}}$, and driver $f(s, y', y, z, k)ds + dA_s$. We have $f(s, y', y, z, k)ds +$

$dA_s \geq f(s, y', y, z, k)ds$ and $Y_{\hat{\tau}} \geq \xi_{\hat{\tau}}$ a.s. Under the assumption (V.14), we apply the strict comparison theorem for mean-field BSDEs with jumps to Y and $X(\xi_{\hat{\tau}}, \hat{\tau})$, then (V.16) follows. □

When there is more regularity on the payoff of the process ξ, we provide the minimal and the maximal S- optimal stopping times. They provide an interval for the stopping time: if the position is closed before the minimal stopping time or after the maximal stopping time, the risk is highest. There are no monotonicity guarantees within the interval. However, under a left regularity condition on the obstacle, an optimal stopping time (within this interval) can be obtained as the limit of τ_S^ε as ε tends to 0.

Theorem 2.2. *Let $(\xi_t, 0 \leq t \leq T)$ be a RCLL process in $\mathcal{S}^2$, assumed to be l.u.s.c. along stopping times, and let f be a Lipschitz driver satisfying Assumption 3.2. Let $S \in \mathbb{T}_0$.*

(i) The stopping time $\tilde{\tau}_S$ defined by

$$\tilde{\tau}_S := \lim_{\varepsilon \downarrow 0} \uparrow \tau_S^\varepsilon,$$

with τ_S^ε given in (VII.7)*, is an S-optimal stopping time.*

(ii) The stopping time τ_S^* defined by

$$\tau_S^* := \inf\{u \geq S;\, Y_u = \xi_u\}$$

is an S-optimal stopping time and we have

$$Y_s = X_s(\xi_{\tau_S^*}, \tau_S^*), \quad S \leq s \leq \tau_S^* \quad \text{a.s.}$$

We also have $\tau_S^* \geq \tilde{\tau}_S$ a.s.

(iii) The stopping time $\bar{\tau}_S$ defined by

$$\bar{\tau}_S := \inf\{u \geq S;\, A_u - A_S > 0\}$$

is an S-optimal stopping time.

(iv) Suppose moreover that in Assumption 3.1, for all x, π, l_1, l_2, we have

$$\theta_t^{x', x, z, l^1, l^2}(u) > -1 \quad dt \otimes dP - \text{a.s.} \tag{V.17}$$

Then, $\tau_S^* = \tilde{\tau}_S$ a.s. Moreover τ_S^* is the minimal and $\bar{\tau}_S$ is the maximal S-optimal stopping time.

Proof. (i) By letting ε tend to 0 in inequality (V.13), we get

$$Y_S \leq \limsup_{\varepsilon \downarrow 0} X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \quad \text{a.s.} \tag{V.18}$$

For each ω such that the map $\varepsilon \mapsto \tau_S^\varepsilon(\omega)$ from $\mathbb{R}_+^* \to [0, T]$ is constant for ε sufficiently small, we have

$$\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon}(\omega) = \xi_{\tilde{\tau}_S}(\omega).$$

Moreover, since the process (ξ_t) is left-limited, for almost every ω such that for each $\varepsilon > 0$,

$\tau_S^\varepsilon(\omega) < \hat{\tau}_S(\omega)$, we have

$$\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon}(\omega) = \xi_{\tilde{\tau}_S^-}(\omega).$$

Hence, for almost every ω, $\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon}(\omega)$ exists. The continuity property of mean-field BSDEs with respect to terminal conditions (see Prop.4.4), implies

$$\lim_{\varepsilon \downarrow 0} X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) = X_S(\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon}, \tilde{\tau}_S) \quad \text{a.s.} \tag{V.19}$$

Now, the l.u.s.c. property of the obstacle along stopping times yields $\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon} \leq \xi_{\tilde{\tau}_S}$ a.s. By the comparison theorem, it follows that

$$X_S(\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon}, \tilde{\tau}_S) \leq X_S(\xi_{\tilde{\tau}_S}, \tilde{\tau}_S) \quad \text{a.s.}$$

Hence, by (V.18) and (V.19), we get $Y_S \leq X_S(\xi_{\tilde{\tau}_S}, \tilde{\tau}_S)$ a.s. By using the characterization of Y_S as the value function of the optimal stopping time problem (V.6), we get

$$Y_S = X_S(\xi_{\tilde{\tau}_S}, \tilde{\tau}_S) \quad \text{a.s.} \tag{V.20}$$

Thus, $\tilde{\tau}_S$ is an S-optimal stopping time.

(ii) The right continuity of (Y_t) and (ξ_t) ensures that $Y_{\tau_S^*} = \xi_{\tau_S^*}$ a.s. By definition of τ_S^*, we have that almost surely on $[S, \tau_S^*[$, $Y_t > \xi_t$ and hence the process A is constant on $[S, \tau_S^*[$ and even on $[S, \tau_S^*]$ because A is continuous (see Theorem 1.2). We derive that $(Y_s, S \leq s \leq \tau_S^*)$ is the solution of the mean-field BSDE associated with terminal time τ_S^*, terminal condition $\xi_{\tau_S^*}$ and driver f, that is, $Y_s = X_s(\xi_{\tau_S^*}, \tau_S^*)$, $S \leq s \leq \tau_S^*$ a.s. Hence, τ_S^* is an S-optimal stopping time.
Furthermore, for each $\varepsilon > 0$, $\tau_S^\varepsilon \leq \tau_S^*$ a.s. By letting ε tend to 0, we get $\tilde{\tau}_S \leq \tau_S^*$. a.s.

(iii) From the definition of $\bar{\tau}_S$, and the continuity of A, we have $A_{\bar{\tau}_S} - A_S = 0$ a.s. Hence

$$Y_s = X_s(Y_{\bar{\tau}_S}, \bar{\tau}_S), \quad S \leq s \leq \bar{\tau}_S \quad \text{a.s.}$$

Also, we have a.s. for all $t > \bar{\tau}_S$, $A_t > A_{\bar{\tau}_S} = A_S$. Since A increases only on the set $\{Y_\cdot = \xi_\cdot\}$, it follows that $Y_{\bar{\tau}_S} = \xi_{\bar{\tau}_S}$. Hence $Y_S = X_S(\xi_{\bar{\tau}_S}, \bar{\tau}_S)$ a.s. In other words, $\bar{\tau}_S$ is S-optimal.

(iv) Let $\hat{\tau}$ be an S-optimal stopping time. By the strict comparison theorem for non reflected mean-field BSDEs (Theorem 1.3 or Proposition 2.1), we have $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ a.s. Hence, by definition of τ_S^*, we have $\hat{\tau} \geq \tau_S^*$ a.s. Thus, $\tilde{\tau}_S \geq \tau_S^*$ a.s.. By (ii), $\tilde{\tau}_S \leq \tau_S^*$ which implies $\tilde{\tau}_S = \tau_S^*$ a.s. We also have proven that τ_S^* is the minimal S-optimal stopping time.

Let us now show that $\bar{\tau}_S$ is the maximal S-optimal stopping time. If $\hat{\tau}$ is S-optimal, by the optimality criterium, $A_{\hat{\tau}_S} - A_S = 0$ a.s. which implies $\hat{\tau} \leq \bar{\tau}_S$ a.s. □

3 Optimization principles for Reflected Mean-Field BSDEs

Now, we denote by (Y, Z, k) the solution of the mean-field Reflected BSDE associated to obstacle (ξ_t) and driver f, and by $(Y^\alpha, Z^\alpha, k^\alpha)$ the solution of the mean-field Reflected BSDE associated with obstacle (ξ_t) and driver f^α.

For each $\tau \in \mathbb{T}_0$ and $\zeta \in L^2(\mathcal{F}_\tau)$, let $(X(\zeta,\tau), Z(\zeta,\tau), l(\zeta,\tau))$ be the solution of the mean-field BSDE associated with driver f, terminal conditions ζ, τ, and $(X^\alpha(\zeta,\tau), Z^\alpha(\zeta,\tau), l^\alpha(\zeta,\tau))$ be the solution of the mean-field BSDE associated with driver f^α and terminal conditions ζ, τ.

Let $S \in \mathbb{T}_0$. We consider the following optimization problem

$$\operatorname*{ess\,inf}_\alpha Y_S^\alpha. \tag{V.21}$$

We first state a characterization of the value function of this problem as well as an existence result, which generalizes a result established in [12] to the case of mean-field case.

Proposition 3.1 (Optimization principle for RBSDEs I). *Suppose that*

(i) *For each* $\alpha \in \mathcal{A}$, $f(t,y',y,z,k) \le f^\alpha(t,y',y,z,k)$, *for all* $(y',y,z,k) \in \mathbb{R}^3 \times \mathcal{L}_\nu^2$; $dt \otimes dP - a.s.$

(ii) *There exists* $\bar\alpha \in \mathcal{A}$ *such that*

$$f(t,F(t,Y_t(\cdot)),Y_t,Z_t,k_t) = f^{\bar\alpha}(t,F(t,Y_t(\cdot)),Y_t,Z_t,k_t),\quad 0 \le t \le T,\ \ dt \otimes dP-\text{a.s.} \tag{V.22}$$

Then, for each $S \in \mathbb{T}_0$,

$$Y_S = \operatorname*{ess\,inf}_\alpha Y_S^\alpha = Y_S^{\bar\alpha} \quad \text{a.s.} \tag{V.23}$$

Proof. We apply the same argument as in the Proof of Proposition 3.1 with only the replacement of the comparison theorem for mean-field RBSDEs (see Theorem 1.3) □

Using an estimate on mean-field RBSDEs (see (V.38)), we derive a similar characterization of the value function of the problem (V.21) under weaker hypotheses.

Proposition 3.2 (Optimization principle for RBSDEs II). *Suppose that the drivers* f^α, $\alpha \in \mathcal{A}$ *satisfy* $f \le f^\alpha$ *and are equi-Lipschitz with constant* C.
Suppose moreover that for each $\eta > 0$ *, there exists* $\alpha^\eta \in \mathcal{A}$ *such that*

$$f(t,F(t,Y_t(\cdot)),Y_t,Z_t,k_t) \ge f^{\bar\alpha}(t,F(t,Y_t(\cdot)),Y_t,Z_t,k_t) - \eta,\quad 0 \le t \le T,\ dP \otimes dt - \text{a.s.} \tag{V.24}$$

Then, for each $S \in \mathbb{T}_0$, *we have*

$$Y_S = \operatorname*{ess\,inf}_\alpha Y_S^\alpha \quad \text{a.s.} \tag{V.25}$$

Proof. Since $f \le f^\alpha$, we have $Y \le Y^\alpha$ a.s. for each $\alpha \in \mathcal{A}$. It follows that for each $S \in \mathbb{T}_0$, we have $Y_S \le \operatorname{ess\,inf}_\alpha Y_S^\alpha$ a.s. Since Assumption (V.24) holds, by using estimate (V.38), with $\eta = \frac{1}{C^2}$ and $\beta = 5C^2 + 2C$, we derive that there exists a constant $K \ge 0$, which depends only on C and T, such that, for each $\eta > 0$ and for each $S \in \mathbb{T}_0$,

$$Y_S + K\,\eta \ge Y_S^{\alpha^\eta} \ge \operatorname*{ess\,inf}_\alpha Y_S^\alpha \quad \text{a.s.}$$

Equality (V.25) thus follows. □

Remark 3.3. *Propositions 3.1 and 3.2 can be seen as* verification theorems *in the following sense: let $f^\alpha, \alpha \in \mathcal{A}$ be a family of drivers. If we are given a driver $f \leq f^\alpha, \alpha \in \mathcal{A}$ satisfying* (V.22) *or* (V.24)*, then the solution Y of the mean-field RBSDE with driver f coincides with the value function of the optimization problem* (V.21). *Under some conditions on the drivers f^α, f can be explicitly defined in terms of the family $f^\alpha, \alpha \in \mathcal{A}$; see e.g. Section 2.*

By applying the strict comparison theorem for reflected mean-field BSDEs (see Theorem 1.4), we provide now some necessary optimality conditions at a given time $S \in \mathbb{T}_0$.

Proposition 3.4 (Necessary optimality conditions)**.** *Suppose that the assumptions of Proposition 3.1 or Proposition 3.2 hold. Let $\hat{\alpha} \in \mathcal{A}$, and suppose that in Assumption 3.1 the coefficient $\theta^{\hat{\alpha}}$ corresponding to driver $f^{\hat{\alpha}}$ satisfies $\theta_t^{\alpha,x',x,z,l^1,l^2}(u) > -1$, for each $x', x, \pi, \ell_1, \ell_2$. Let $S \in \mathbb{T}_0$. Suppose that $\hat{\alpha}$ is S-optimal, i.e.*

$$ess \inf_{\alpha} Y_S^\alpha = Y_S^{\hat{\alpha}} \quad \text{a.s.} \tag{V.26}$$

(i) *Suppose ξ is l.u.s.c. along stopping times. Let $\tau_S^* := \inf\{t \geq S,\ Y_t = \xi_t\}$. Then*

$$Y_{\tau_S^*}^{\hat{\alpha}} = \xi_{\tau_S^*} \text{ a.s.}\,;\, f(t, F(t, Y_t(\cdot)), Y_t, Z_t, k_t) = f^{\hat{\alpha}}(t, F(t, Y_t(\cdot)), Y_t, Z_t, k_t),\ S \leq t \leq \tau_S^*,\ dP \otimes dt-a.s. \tag{V.27}$$

(ii) *Consider the case when ξ is not supposed to be l.u.s.c. along stopping times.*
For each $\varepsilon > 0$, let $\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\}$. Then for each $\varepsilon > 0$,

$$Y_{\tau_S^\varepsilon}^{\hat{\alpha}} \leq \xi_{\tau_S^\varepsilon} + \varepsilon \text{ a.s.}\,;\, f(t, F(t, Y_t(\cdot)), Y_t, Z_t, k_t) = f^{\hat{\alpha}}(t, F(t, Y_t(\cdot)), Y_t, Z_t, k_t)),\ S \leq t \leq \tau_S^\varepsilon,\ dP \otimes dt-\text{a.s.} \tag{V.28}$$

Proof. By Proposition 3.1 or Proposition 3.2, we have $Y_S = ess \inf_\alpha Y_S^\alpha$ a.s. From equality (V.26), it follows that $Y_S = Y_S^{\hat{\alpha}}$ a.s.

1. Since $Y \leq Y^{\hat{\alpha}}$, it follows that $\tau_S^* \leq \tau_S^{\hat{\alpha},*}$ where $\tau_S^{\hat{\alpha},*} := \inf\{t \geq S,\ Y_t^{\hat{\alpha}} = \xi_t\}$. By the strict comparison Theorem 1.4 1. applied to $\xi^1 = \xi^2 = \xi$, $f^1 = f$, $f^2 = f^{\hat{\alpha}}$, $Y^1 = Y$, $Y^2 = Y^{\hat{\alpha}}$, since $Y_S = Y_S^{\hat{\alpha}}$ a.s. , we derive equalities (V.27).

2. Let

$$\tau_S^{\hat{\alpha},\varepsilon} := \inf\{t \geq S,\ Y_t^{\hat{\alpha}} \leq \xi_t + \varepsilon\}. \tag{V.29}$$

Since $Y \leq Y^{\hat{\alpha}}$, it follows that for each $\varepsilon > 0$, we have $\tau_S^\varepsilon \leq \tau_S^{\hat{\alpha},\varepsilon}$ a.s. By the strict comparison Theorem 1.4 2. applied to $\xi^1 = \xi^2 = \xi$, $f^1 = f$, $f^2 = f^{\hat{\alpha}}$, $Y^1 = Y$, $Y^2 = Y^{\hat{\alpha}}$, since $Y_S = Y_S^{\hat{\alpha}}$ a.s. , we derive (V.28). □

We now provide sufficient conditions of optimality at a given time $S \in \mathbb{T}_0$, which are weaker than those made in Proposition 3.1.

Proposition 3.5 (Sufficient optimality conditions)**.**
Suppose that for each $\alpha \in \mathcal{A}$, $f \leq f^\alpha$. Let $\hat{\alpha} \in \mathcal{A}$ and $S \in \mathbb{T}_0$.

(i) *Suppose ξ is l.u.s.c. along stopping times.*
If equalities (V.27) *hold, then $\hat{\alpha}$ is S-optimal, that is, $ess \inf_\alpha Y_S^\alpha = Y_S^{\hat{\alpha}}$ a.s.*

(ii) *Consider the case when ξ is not supposed to be l.u.s.c. along stopping times.*
If for each $\varepsilon > 0$, conditions (V.28) *hold, then $\hat{\alpha}$ is S-optimal.*

In both cases, we get $Y_S = ess\inf_\alpha Y_S^\alpha$ *a.s.*

Proof. For all α, since $f \leq f^\alpha$, we have $Y_S \leq Y_S^\alpha$ a.s. and thus $Y_S \leq \operatorname{ess\,inf}_\alpha Y_S^\alpha$ a.s.

1. Since $Y \leq Y^{\hat{\alpha}}$, it follows that $\tau_S^* \leq \tau_S^{\hat{\alpha},*}$ where $\tau_S^{\hat{\alpha},*} := \inf\{t \geq S,\ Y_t^{\hat{\alpha}} = \xi_t\}$. Suppose that equalities (V.27) hold. Then, by the optimality of τ_S^* for Y_S, we have

$$Y_t = X_t(\xi_{\tau_S^*}, \tau_S^*), \qquad S \leq t \leq \tau_S^*, \quad \text{a.s.}$$

This with equality (V.27) and the uniqueness result for mean-field BSDEs leads to

$$Y_t = X_t(\xi_{\tau_S^*}, \tau_S^*) = X_t^{\hat{\alpha}}(\xi_{\tau_S^*}, \tau_S^*) = X_t^{\hat{\alpha}}(Y_{\tau_S^*}^{\hat{\alpha}}, \tau_S^*), \qquad S \leq t \leq \tau_S^*, \quad \text{a.s.}\,,$$

Moreover, according to the previous equalities, $X_t^{\hat{\alpha}}(Y_{\tau_S^*}^{\hat{\alpha}}, \tau_S^*) = Y_t \geq \xi_t,\ S \leq t \leq \tau_S^*$ a.s.
By the uniqueness result for mean-field RBSDEs, we get

$$Y_t = X_t^{\hat{\alpha}}(Y_{\tau_S^*}^{\hat{\alpha}}, \tau_S^*) = Y_t^{\hat{\alpha}}, \qquad S \leq t \leq \tau_S^*,\ \text{a.s.}$$

By taking $t = S$, we get $Y_S = \operatorname{ess\,inf}_\alpha Y_S^\alpha = Y_S^{\hat{\alpha}}$ a.s.

2. Since $Y \leq Y^{\hat{\alpha}}$, for each $\varepsilon > 0$, we have $\tau_S^\varepsilon \leq \tau_S^{\hat{\alpha},\varepsilon}$ a.s. where $\tau_S^{\hat{\alpha},\varepsilon}$ is defined in (V.29). Let us now show that $Y_S \geq Y_S^{\hat{\alpha}}$ a.s. By arguments in the Proof of Theorem 2.1, we have

$$Y_t = X_t(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon), \qquad S \leq t \leq \tau_S^\varepsilon,\ \text{a.s.}$$

Hence, using equality (V.28), we derive that

$$Y_t = X_t(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) = X_t^{\hat{\alpha}}(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon), \qquad S \leq t \leq \tau_S^\varepsilon,\ \text{a.s.}\,.$$

By the comparison theorem for non reflected mean-field BSDEs and inequality $Y_{\tau_S^\varepsilon} \geq \xi_{\tau_S^\varepsilon}$ a.s. , we have

$$Y_t = X_t^{\hat{\alpha}}(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \geq X_t^{\hat{\alpha}}(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon), \qquad S \leq t \leq \tau_S^\varepsilon, \quad \text{a.s.}$$

Now, by A priori estimates (see Proposition 4.2), we have

$$Y_S \geq X_S^{\hat{\alpha}}(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \geq X_S^{\hat{\alpha}}(\xi_{\tau_S^\varepsilon} + \varepsilon\,, \tau_S^\varepsilon) - \varepsilon e^{\frac{\beta T}{2}} \quad \text{a.s.}$$

with $\beta = 3C^2 + 4C$, where C is the Lipschistz constant of $f^{\hat{\alpha}}$. Since by assumption, $\xi_{\tau_S^\varepsilon} + \varepsilon \geq Y_{\tau_S^\varepsilon}^{\hat{\alpha}}$ a.s. , the comparison theorem for non reflected mean-field BSDEs (Proposition 1.2) yields that

$$Y_S + \varepsilon e^{\frac{\beta T}{2}} \geq X_S^{\hat{\alpha}}(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) \geq X_S^{\hat{\alpha}}(Y_{\tau_S^\varepsilon}^{\hat{\alpha}}\,, \tau_S^\varepsilon) \quad \text{a.s.}$$

By arguments in the Proof of Theorem 2.1,, the non decreasing process associated with $Y_\cdot^{\hat{\alpha}}$ is constant on $[S, \tau_S^{\hat{\alpha},\varepsilon}]$ and hence on $[S, \tau_S^\varepsilon]$, because $\tau_S^\varepsilon \leq \tau_S^{\hat{\alpha},\varepsilon}$ a.s. Thus, $(Y_t^{\hat{\alpha}}, S \leq t \leq \tau_S^\varepsilon)$ is the solution of the non reflected BSDE associated with driver $f^{\hat{\alpha}}$, terminal conditions $(\tau_S^\varepsilon, Y_{\tau_S^\varepsilon}^{\hat{\alpha}})$. We thus get

$$X_S^{\hat{\alpha}}(Y_{\tau_S^\varepsilon}^{\hat{\alpha}}, \tau_S^\varepsilon) = Y_S^{\hat{\alpha}} \quad \text{a.s.}$$

Consequently, for each $\varepsilon > 0$, we have $Y_S + \varepsilon e^{\frac{\beta T}{2}} \geq Y_S^{\hat{\alpha}}$ a.s. , and hence, $Y_S \geq Y_S^{\hat{\alpha}}$ a.s. We thus have $Y_S = \operatorname{ess\,inf}_\alpha Y_S^\alpha = Y_S^{\hat{\alpha}}$ a.s. , which provides the desired result. $\square$

4 Appendix

Lemma 4.1. *([12] Proposition A.1) Let $T > 0$ and let $\xi \in \mathcal{S}^2$. Let f^1 be a Lipschitz driver with Lipschitz constant C and let f^2 be a driver. For $i = 1, 2$, let (Y^i, Z^i, k^i, A^i) be a solution of the RBSDE (V.1) associated to terminal time T, driver f^i and obstacle ξ. For s in $[0, T]$, denote $\bar{Y}_s := Y^1_s - Y^2_s$, $\bar{Z}_s := Z^1_s - Z^2_s$, $\bar{k}_s := k^1_s - k^2_s$, and $\bar{f}(s) := f^1(s, Y^2_s, Z^2_s, k^2_s) - f^2(s, Y^2_s, Z^2_s, k^2_s)$. Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 2C$. If $\eta \leq \frac{1}{C^2}$, then, for each $t \in [0, T]$, we have*

$$e^{\beta t}\bar{Y}_t^2 \leq \eta\, E[\int_t^T e^{\beta s}\bar{f}(s)^2 ds \mid \mathcal{F}_t] \quad \text{a .s.} \tag{V.30}$$

$$\|\bar{Y}\|^2_\beta \leq T\eta\|\bar{f}\|^2_\beta. \tag{V.31}$$

Also, if $\eta < \frac{1}{C^2}$, we then have

$$\|\bar{Z}\|^2_\beta + \|\bar{k}\|^2_{\nu,\beta} \leq \frac{\eta}{1 - \eta C^2}\|\bar{f}\|^2_\beta. \tag{V.32}$$

Lemma 4.2. *(Apriori estimates for mean-field BSDE with jumps) Let $T > 0$ and let $\xi^1, \xi^2 \in \mathcal{S}^2$. Let f^1 be a Lipschitz driver with Lipschitz constant C and let f^2 be a driver. For $i = 1, 2$, let (Y^i, Z^i, k^i) be a solution of the Mean-field BSDE (IV.5) associated to terminal time T, driver f^i, Mean-field operator F and terminal condition ξ^i. For s in $[0, T]$, denote $\bar{Y}_s := Y^1_s - Y^2_s$, $\bar{Z}_s := Z^1_s - Z^2_s$, $\bar{k}_s := k^1_s - k^2_s$, and $\bar{f}(s) := f^1(s, F(s, Y^2_s(\cdot)), Y^2_s, Z^2_s, k^2_s) - f^2(s, F(s, Y^2_s(\cdot))Y^2_s, Z^2_s, k^2_s)$.
Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 4C$. If $\eta \leq \frac{1}{C^2}$, then, for each $t \in [0, T]$, we have*

$$e^{\beta t}\bar{Y}_t^2 \leq \mathbb{E}[e^{\beta T}[\bar{\xi}^2] \mid \mathcal{F}_t] + \eta\, E[\int_t^T e^{\beta s}\bar{f}(s)^2 ds \mid \mathcal{F}_t] \quad \text{a .s.} \tag{V.33}$$

$$\|\bar{Y}\|^2_\beta \leq T[e^{\beta T}[\bar{\xi}^2] + \eta\|\bar{f}\|^2_\beta]. \tag{V.34}$$

Also, if $\eta < \frac{1}{C^2}$, we then have

$$\|\bar{Z}\|^2_\beta + \|\bar{k}\|^2_{\nu,\beta} \leq \frac{1}{1 - \eta C^2}\left[e^{\beta T}[\bar{\xi}^2] + \|\bar{f}\|^2_\beta\right]. \tag{V.35}$$

Proof. From Itô's formula applied to the semimartingale $e^{\beta s}\bar{Y}_s^2$ between t and T, it follows

$$\begin{aligned}
&e^{\beta t}\bar{Y}_t^2 + \beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}\bar{Z}_s^2 ds + \int_t^T e^{\beta s}\|\bar{k}_s\|^2_\nu ds \\
&\quad = 2\int_t^T e^{\beta s}\bar{Y}_s[f^1(s, F(s, Y^1_s(\cdot), Y^1_s, Z^1_s, k^1_s) - f^2(s, F(s, Y^2_s(\cdot)), Y^2_s, Z^2_s, k^2_s)]ds \\
&\qquad - 2\int_t^T e^{\beta s}\bar{Y}_s\bar{Z}_s dW_s - \int_t^T e^{\beta s}\int_{\mathbb{R}^*}(2\bar{Y}_{s^-}\bar{k}_s(u) + \bar{k}_s(u)^2)d\tilde{N}(du, dt)
\end{aligned}$$

Taking the conditional expectation given $\mathcal{F}_t$, we get

$$e^{\beta t}\bar{Y}_t^2 + \mathbb{E}\left[\beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2)ds \mid \mathcal{F}_t\right]$$
$$\leq 2\mathbb{E}\left[\int_t^T e^{\beta s}\bar{Y}_s[f^1(s, F(s, Y_s^1(\cdot), Y_s^1, Z_s^1, k_s^1) - f^2(s, F(s, Y_s^2(\cdot)), Y_s^2, Z_s^2, k_s^2)]ds \mid \mathcal{F}_t\right]. \tag{V.36}$$

Moreover,

$$\begin{aligned}
&|[f^1(s, F(s, Y_s^1(\cdot), Y_s^1, Z_s^1, k_s^1) - f^2(s, F(s, Y_s^2(\cdot)), Y_s^2, Z_s^2, k_s^2)]| \\
&\leq |f^1(s, F(s, Y_s^1(\cdot)), Y_s^1, Z_s^1, k_s^1) - f^1(s, F(s, Y_s^2(\cdot)), Y_s^2, Z_s^2, k_s^2)| + |\bar{f}_s| \\
&\leq C(\mathbb{E}|\bar{Y}_s|^2)^{\frac{1}{2}} + C(|\bar{Y}_s| + |\bar{Z}_s| + \|\bar{k}_s\|_\nu) + |\bar{f}_s|.
\end{aligned}$$

Now, for all real numbers y', y, z, k, f and $\varepsilon > 0$
$2y(Cz + Ck + f) \leq \frac{y^2}{\varepsilon^2} + \varepsilon^2(Cz + Ck + f)^2 \leq \frac{y^2}{\varepsilon^2} + 3\varepsilon^2(C^2z^2 + C^2k^2 + f^2)$.
And $\mathbb{E}[2\bar{Y}_s(\mathbb{E}|\bar{Y}_s|^2)^{\frac{1}{2}}] \leq \mathbb{E}[|\bar{Y}_s|^2 + \mathbb{E}|\bar{Y}_s|^2] = 2\mathbb{E}|\bar{Y}_s|^2$

Hence, we get

$$e^{\beta t}\bar{Y}_t^2 + E\left[\beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2)ds \mid \mathcal{F}_t\right]$$
$$\leq E\left[e^{\beta T}(\xi_1 - \xi_2)^2 + (4C + \frac{1}{\varepsilon^2})\int_t^T e^{\beta s}\bar{Y}_s^2 ds + 3C^2\varepsilon^2\int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2)ds \mid \mathcal{F}_t\right]$$
$$+ 3\varepsilon^2 E\left[\int_t^T e^{\beta s}\bar{f}_s^2 ds \mid \mathcal{F}_t\right]. \tag{V.37}$$

Let us make the change of variable $\eta = 3\epsilon^2$. Then, for each $\beta, \eta > 0$ chosen as in the proposition, these inequalities lead to (V.33). We obtain the first inequality of (V.34) by integrating (V.33). Then (V.35) follows from inequality (V.37). □

Apriori estimates for Mean-field RBSDE

Lemma 4.3. *(Apriori estimates for Mean-field RBSDE with jumps) Let $T > 0$ and let $\xi \in \mathcal{S}^2$. Let f^1 be a Lipschitz driver with Lipschitz constant C and let f^2 be a driver. For $i = 1, 2$, let (Y^i, Z^i, k^i, A^i) be a solution of the RBSDE V.2 associated to terminal time T, driver f^i, Mean-field operator F and obstacle ξ. For s in $[0, T]$, denote $\bar{Y}_s := Y_s^1 - Y_s^2$, $\bar{Z}_s := Z_s^1 - Z_s^2$, $\bar{k}_s := k_s^1 - k_s^2$, and*
$\bar{f}(s) := f^1(s, F(s, Y_s^2(\cdot)), Y_s^2, Z_s^2, k_s^2) - f^2(s, F(s, Y_s^2(\cdot))Y_s^2, Z_s^2, k_s^2)$.

Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 4C$. If $\eta \leq \frac{1}{C^2}$, then, for each $t \in [0, T]$, we have

$$e^{\beta t}\bar{Y}_t^2 \leq \eta\, E[\int_t^T e^{\beta s}\bar{f}(s)^2 ds \mid \mathcal{F}_t] \quad \text{a .s.} \tag{V.38}$$

$$\|\bar{Y}\|_\beta^2 \leq T\eta\|\bar{f}\|_\beta^2. \tag{V.39}$$

Also, if $\eta < \frac{1}{C^2}$, we then have

$$\|\bar{Z}\|_\beta^2 + \|\bar{k}\|_{\nu,\beta}^2 \leq \frac{\eta}{1-\eta C^2}\|\bar{f}\|_\beta^2. \tag{V.40}$$

Proof. From Itô's formula applied to the semimartingale $e^{\beta s}\bar{Y}_s^2$ between t and T, it follows

$$\begin{aligned}
&e^{\beta t}\bar{Y}_t^2 + \beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}\bar{Z}_s^2 ds + \int_t^T e^{\beta s}\|\bar{k}_s\|_\nu^2 ds + \sum_{t<s\leq T} e^{\beta s}(\Delta A_s^1 - \Delta A_s^2)^2 \\
&= 2\int_t^T e^{\beta s}\bar{Y}_s[f^1(s, F(s, Y_s^1(\cdot), Y_s^1, Z_s^1, k_s^1) - f^2(s, F(s, Y_s^2(\cdot)), Y_s^2, Z_s^2, k_s^2)]ds \\
&\quad - 2\int_t^T e^{\beta s}\bar{Y}_s\bar{Z}_s dW_s - \int_t^T e^{\beta s}\int_{\mathbb{R}^*}(2\bar{Y}_{s^-}\bar{k}_s(u) + \bar{k}_s(u)^2)d\tilde{N}(du, dt) \\
&\quad + 2\int_t^T e^{\beta s}\overline{Y}_{s^-}dA_s^1 - 2\int_t^T e^{\beta s}\overline{Y}_{s^-}dA_s^2
\end{aligned} \tag{V.41}$$

Now, we have a.s.

$$\overline{Y}_s dA_s^{1,c} = (Y_s^1 - \xi_s)dA_s^{1,c} - (Y_s^2 - \xi_s)dA_s^{1,c} = -(Y_s^2 - \xi_s)dA_s^{1,c} \leq 0$$

and by symmetry, $\overline{Y}_s dA_s^{2,c} \geq 0$ a.s. Also, we have a.s.

$$\overline{Y}_{s^-}\Delta A_s^{1,d} = (Y_{s^-}^1 - \xi_{s^-})\Delta A_s^{1,d} - (Y_{s^-}^2 - \xi_{s^-})\Delta A_s^{1,d} = -(Y_{s^-}^2 - \xi_{s^-})\Delta A_s^{1,d} \leq 0$$

and $\overline{Y}_{s^-}\Delta A_s^{2,d} \geq 0$ a.s. Consequently, the two last terms of the r.h.s. of (V.41) are non positive. Taking the conditional expectation given $\mathcal{F}_t$, we get

$$\begin{aligned}
&e^{\beta t}\bar{Y}_t^2 + \mathbb{E}\left[\beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2)ds \mid \mathcal{F}_t\right] \\
&\quad \leq 2\mathbb{E}\left[\int_t^T e^{\beta s}\bar{Y}_s[f^1(s, F(s, Y_s^1(\cdot), Y_s^1, Z_s^1, k_s^1) - f^2(s, F(s, Y_s^2(\cdot)), Y_s^2, Z_s^2, k_s^2)]ds \mid \mathcal{F}_t\right].
\end{aligned} \tag{V.42}$$

Moreover,

$$\begin{aligned}
&|[f^1(s, F(s, Y_s^1(\cdot), Y_s^1, Z_s^1, k_s^1) - f^2(s, F(s, Y_s^2(\cdot)), Y_s^2, Z_s^2, k_s^2)]| \\
&\leq |f^1(s, F(s, Y_s^1(\cdot), Y_s^1, Z_s^1, k_s^1) - f^1(s, F(s, Y_s^2(\cdot)), Y_s^2, Z_s^2, k_s^2)| + |\bar{f}_s| \\
&\qquad \leq C(\mathbb{E}|\bar{Y}_s|^2)^{\frac{1}{2}} + C(|\bar{Y}_s| + |\bar{Z}_s| + \|\bar{k}_s\|_\nu) + |\bar{f}_s|.
\end{aligned}$$

Now, for all real numbers y', y, z, k, f and $\varepsilon > 0$
$2y(Cz + Ck + f) \leq \frac{y^2}{\varepsilon^2} + \varepsilon^2(Cz + Ck + f)^2 \leq \frac{y^2}{\varepsilon^2} + 3\varepsilon^2(C^2z^2 + C^2k^2 + f^2)$.
And $\mathbb{E}[2\bar{Y}_s(\mathbb{E}|\bar{Y}_s|^2)^{\frac{1}{2}}] \leq \mathbb{E}[|\bar{Y}_s|^2 + \mathbb{E}|\bar{Y}_s|^2] = 2\mathbb{E}|\bar{Y}_s|^2$

Hence, we get

$$\begin{aligned} e^{\beta t}\bar{Y}_t^2 + E\left[\beta \int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2) ds \mid \mathcal{F}_t\right] \\ \leq E\left[(4C + \frac{1}{\varepsilon^2}) \int_t^T e^{\beta s}\bar{Y}_s^2 ds + 3C^2\varepsilon^2 \int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2) ds \mid \mathcal{F}_t\right] \\ + 3\varepsilon^2 E\left[\int_t^T e^{\beta s}\bar{f}_s^2 ds \mid \mathcal{F}_t\right]. \end{aligned} \tag{V.43}$$

Let us make the change of variable $\eta = 3\epsilon^2$. Then, for each $\beta, \eta > 0$ chosen as in the proposition, these inequalities lead to (V.38). We obtain the first inequality of (V.39) by integrating (V.38). Then (V.40) follows from inequality (V.43). □

Proposition 4.4. *(A continuity result) Let $T \in [0, T']$, let $\{\theta^\alpha, \alpha \in R\}$ be a family of stopping time in $\mathbb{T}_{0,T}$, converging a.s. to a stopping time $\theta \in \mathbb{T}_{0,T}$ as α tends to α_0. let $\{\xi^\alpha, \alpha \in R\}$ be a family of random variables such that $\mathbb{E}[esssup_\alpha(\xi^\alpha)^2] < \infty$, and for each α, ξ^α is $\mathcal{F}_{\theta^\alpha}$ measurable. Suppose also that ξ^α converges a.s. to an $\mathcal{F}_{\theta^\alpha}$ measurable random variable ξ as α tends to α_0. Let f be standard driver and F be the Mean-field operator. Let $X^\alpha := X(\xi^\alpha, \theta^\alpha)$; $Z^\alpha := Z(\xi^\alpha, \theta^\alpha)$; $l^\alpha := l(\xi^\alpha, \theta^\alpha)$. Then for each $S \in \mathbb{T}_{0,T}$, the random variable X_S^α converges to X_S a.s., and the processes X^α converges to X in $S^{2,T}$.*

Proof. Let (X, Z, l) be the solution associated with mean-field BSDE with terminal time T, terminal condition ξ, driver $f(t, x', x, z, l)1_{t\leq\theta}$ and mean-field operator F. Also, let $(X^\alpha, Z^\alpha, l^\alpha)$ be the solution associated with mean-field BSDE with terminal time T, terminal condition ξ^α, driver $f(t, x', x, z, l)1_{t\leq\theta^\alpha}$ and Mean-field operator F. Applying the estimates (V.33), we get,for each stopping time S,

$$e^{\beta S}(X_S - X_S^\alpha)^2 \leq \mathbb{E}\left[e^{\beta T}(\xi - \xi^\alpha)^2 + \eta \int_{(\theta^\alpha \wedge \theta)\vee S}^{(\theta^\alpha \vee \theta)\wedge S} e^{\beta T} f(s, F(s, X_s), X_s, Z_s, l_s)^2 ds \mid \mathcal{F}_S\right]$$

with β and η as in Proposition 4.2. By the assumptions and the Lebesgue theorem, we conclude that X_S^α converges to X_S a.s. Moreover, by (V.34) the processes X^α converges to X in $S^{2,T}$.
□

Chapter VI

Optimal stopping for Global risk measures in the case of multiple priors

---

1 Robust optimal stopping problem

We now consider the optimal stopping problem when there is ambiguity on the risk-measure modeling. Let $\{f^\alpha, \alpha \in \mathcal{A}\}$ be a given family of Lipschitz drivers satisfying Assumption (3.1). For each $\alpha \in \mathcal{A}$, let ρ^α be the risk measure induced by the mean-field BSDE with driver f^α. That is, as defined in (IV.19), for each terminal time $\tau \in \mathbb{T}_0$ and position $\zeta \in L^2(\mathcal{F}_\tau)$,

$$\rho_t^\alpha(\zeta, \tau) = -X_t^\alpha(\zeta, \tau), \quad 0 \leq t \leq T,$$

where $X_t^\alpha(\zeta, \tau)$ denotes the solution of the mean-field BSDE associated with driver f^α, terminal condition ζ and terminal time τ. We consider an agent who is averse to ambiguity, and we define her risk measure of position ζ, at each time S in $\mathbb{T}_0$ with $S \leq \tau$ a.s., as the supremum over α of the associated risk-measures $\rho_S^\alpha(\zeta, \tau)$ that is,

$$\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \rho_S^\alpha(\zeta, \tau) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} -X_S^\alpha(\zeta, \tau).$$

Let (ξ_t) be a dynamic position, given by an RCLL adapted process (ξ_t) in $\mathcal{S}^2$. At time $S \in \mathbb{T}_0$, the agent wants to find a stopping time $\tau \in \mathbb{T}_S$ which minimizes her risk measure. At time S, her value function is defined as

$$u(S) := \operatorname*{ess\,inf}_{\tau \in \mathbb{T}_S} \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \rho_S^\alpha(\xi_\tau, \tau). \tag{VI.1}$$

Let $S \in \mathbb{T}_0$. Define the *first value function at time* S as

$$\overline{V}(S) := \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} \operatorname*{ess\,sup}_{\tau \in \mathbb{T}_S} X_S^\alpha(\xi_\tau, \tau), \tag{VI.2}$$

and the *second value function at time* S as

$$\underline{V}(S) := \operatorname*{ess\,sup}_{\tau \in \mathbb{T}_S} \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} X_S^\alpha(\xi_\tau, \tau). \tag{VI.3}$$

Note that $\underline{V}(S) = -u(S)$ a.s.
By definition, we say that there exists a *value function* at time S for the game problem if

$\underline{V}(S) = \overline{V}(S)$ a.s.

Definition 1.1 (S-Saddle point). *Let $S \in \mathbb{T}_0$. A pair $(\hat{\tau}, \hat{\alpha}) \in \mathbb{T}_S \times \mathcal{A}$ is called a S-*saddle point *if*

- $\underline{V}(S) = \overline{V}(S)$ *a.s.,*
- *the essential infimum in (VI.2) is attained at $\hat{\alpha}$,*
- *the essential supremum in (VI.3) is attained at $\hat{\tau}$.*

By classical results, for each $S \in \mathbb{T}_0$, $(\hat{\tau}, \hat{\alpha})$ is a S-saddle point if and only if for each $(\tau, \alpha) \in \mathbb{T}_S \times \mathcal{A}$,

$$X_S^{\hat{\alpha}}(\xi_\tau, \tau) \leq \ X_S^{\hat{\alpha}}(\xi_{\hat{\tau}}, \hat{\tau}) \leq \ X_S^{\alpha}(\xi_{\hat{\tau}}, \hat{\tau}) \text{ a.s.} \tag{VI.4}$$

Note that for each $S \in \mathbb{T}_0$, the inequality $\underline{V}(S) \leq \overline{V}(S)$ a.s. clearly holds. We want to determine when the equality holds, characterize the value function, and address the question of existence of a S-saddle point.

Remark 1.2. *Let $S \in \mathbb{T}_0$. If $(\hat{\tau}, \hat{\alpha})$ is an S-saddle point, then $\hat{\tau}$ and $\hat{\alpha}$ attain respectively the infimum and the supremum in $\underline{V}(S)$ that is,*

$$\underline{V}(S) = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}_S} \operatorname*{ess\,inf}_{\alpha} X_S^{\alpha}(\xi_\tau, \tau) = \operatorname*{ess\,inf}_{\alpha} X_S^{\alpha}(\xi_{\hat{\tau}}, \hat{\tau}) = X_S^{\hat{\alpha}}(\xi_{\hat{\tau}}, \hat{\tau}).$$

Hence, $\hat{\tau}$ is an optimal stopping time for the agent who wants to minimize over stopping times her risk-measure at time S in the case of ambiguity (see (VI.1)*).*
Also, since $\hat{\alpha}$ attains the essential infimum in (VI.2), $\hat{\alpha}$ corresponds at time S to a worst-case scenario. Hence, the robust optimal stopping problem (VI.1) *reduces to a classical optimal stopping problem associated with a worst-case scenario among the possible ambiguity parameters $\alpha \in \mathcal{A}$.*

We relate now the game problem to an optimization problem for mean-field RBSDEs.
Let $(Y^\alpha, Z^\alpha, k^\alpha)$ be the solution of the mean-field RBSDE with obstacle (ξ_t) and driver f^α. For each $\tau \in \mathbb{T}_0$ and $\zeta \in L^2(\mathcal{F}_\tau)$, let $X^\alpha(\zeta, \tau)$ be the solution of the mean-field BSDE with driver f^α and terminal conditions (ζ, τ). By the characterization of mean-field RBSDEs (see Theorem 2.1), for each $S \in \mathbb{T}_0$, we have $Y_S^\alpha = \operatorname{ess\,sup}_{\tau \in \mathbb{T}_S} X_S^\alpha(\xi_\tau, \tau)$ a.s. It follows that

$$\overline{V}(S) = \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} Y_S^\alpha \quad \text{a .s.} \tag{VI.5}$$

Let f be a Lipschitz driver satisfying Assumption 3.1 which is non decreasing in x' and F be the non decreasing Lipschitz operator. Let (Y, Z, k) be the solution of the mean-field RBSDE with obstacle (ξ_t), driver f and mean-field operator F. For each $\tau \in \mathbb{T}_0$ and $\zeta \in L^2(\mathcal{F}_\tau)$, let $X(\zeta, \tau)$ be the solution of the mean-field BSDE with driver f and terminal conditions (ζ, τ).

Theorem 1.1 (Existence and characterization of the common value function - I). *Suppose that the drivers f^α, $\alpha \in \mathcal{A}$ satisfy $f \leq f^\alpha$ and are equi-Lipschitz with constant C. Suppose that there exists $\bar{\alpha}$ such that*

$$f(t, F(t, Y_t(\cdot)), Y_t, Z_t, k_t) = f^{\hat{\alpha}}(t, F(t, Y_t(\cdot)), Y_t, Z_t, k_t), 0 \leq t \leq T, \ \ dt \otimes dP - a.s. \tag{VI.6}$$

Then, there exists a value function, which is characterized as the solution of the mean-field RBSDE with obstacle (ξ_t) *and driver* f*, that is, for each* $S \in \mathbb{T}_0$*, we have*

$$Y_S = \overline{V}(S) = \underline{V}(S) \text{ a.s.}$$

In particular, the minimal risk measure, defined by (VI.1)*, satisfies for each* $S \in \mathbb{T}_0$

$$u(S) = -Y_S \text{ a.s.}$$

Proof. Let $S \in \mathbb{T}_0$. Let us prove that $\overline{V}(S) \leq \underline{V}(S)$ a.s.

By assumption (VI.6) and the optimization principle for mean-field RBSDEs (see Theorem 3.1), we have:

$$\overline{V}(S) = \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} Y_S^{\alpha} = Y_S^{\bar{\alpha}} = Y_S \quad \text{a.s.} \tag{VI.7}$$

Let $\varepsilon > 0$. By a property of τ_S^{ε} (see Proof of Theorem 2.1,), we have

$$Y_t = X_t(Y_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon}),\ S \leq t \leq \tau_S^{\varepsilon}, \quad \text{a.s.}$$

In other terms, (Y_t, Z_t, k_t) is the solution of the Mean-field BSDE associated with driver f and terminal conditions $(Y_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon})$. Now Assumption (VI.6) holds. By an optimization principle for non reflected Mean-field BSDEs, we thus derive that

$$Y_S = X_S(Y_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon}) = \operatorname*{ess\,inf}_{\alpha} X_S^{\alpha}(Y_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon}) \quad \text{a.s.} \tag{VI.8}$$

Using the comparison theorem for non reflected Mean-field BSDEs and the inequality $Y_{\tau_S^{\varepsilon}} \leq \xi_{\tau_S^{\varepsilon}} + \varepsilon$ a.s. , we get

$$Y_S = \operatorname*{ess\,inf}_{\alpha} X_S^{\alpha}(Y_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon}) \leq \operatorname*{ess\,inf}_{\alpha} X_S^{\alpha}(\xi_{\tau_S^{\varepsilon}} + \varepsilon, \tau_S^{\varepsilon}) \quad \text{a.s.} \tag{VI.9}$$

By a priori estimates for non reflected Mean-field BSDEs with jumps (Proposition 4.2), for each $\varepsilon > 0$ and for each $\alpha \in \mathcal{A}$, we have

$$X_S^{\alpha}(\xi_{\tau_S^{\varepsilon}} + \varepsilon, \tau_S^{\varepsilon}) \leq X_S^{\alpha}(\xi_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon}) + \varepsilon e^{\frac{\beta T}{2}} \quad \text{a.s.}\,,$$

with $\beta = 3C^2 + 4C$, where the constant C is equal to the Lipschitz constant common to all the drivers f^{α}, $\alpha \in \mathcal{A}$. By taking the essential infimum over α, we derive that for each $\varepsilon > 0$,

$$\operatorname*{ess\,inf}_{\alpha} X_S^{\alpha}(\xi_{\tau_S^{\varepsilon}} + \varepsilon, \tau_S^{\varepsilon}) \leq \operatorname*{ess\,inf}_{\alpha} X_S^{\alpha}(\xi_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon}) + \varepsilon e^{\frac{\beta T}{2}} \leq \underline{V}(S) + \varepsilon e^{\frac{\beta T}{2}} \quad \text{a.s.}\,,$$

where the last inequality follows from the fact that

$$\underline{V}(S) = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}_S} \operatorname*{ess\,inf}_{\alpha} X_S^{\alpha}(\xi_{\tau}, \tau) \quad \text{a.s.}$$

Using (VI.9), we get $Y_S \leq \underline{V}(S) + \varepsilon e^{\frac{\beta T}{2}}$ a.s. Since $\overline{V}(S) = Y_S$ a.s. (see (VI.7)), it follows that for each $\varepsilon > 0$, we have

$$\overline{V}(S) = Y_S \leq \underline{V}(S) + \varepsilon e^{\frac{\beta T}{2}} \quad \text{a.s.}$$

Hence, $\overline{V}(S) = Y_S \leq \underline{V}(S)$ a.s. Since $\underline{V}(S) \leq \overline{V}(S)$ a.s. , we get $\overline{V}(S) = Y_S = \underline{V}(S)$ a.s. □

Corollary 1.3 (Existence of saddle points)**.** *Suppose that the assumptions of Theorem 1.1 are satisfied and that the obstacle ξ is l.u.s.c. along stopping times. For each $S \in \mathbb{T}_0$, let*

$$\tau_S^* := \inf\{u \geq S;\, Y_u = \xi_u\}.$$

Then, $(\tau_S^*, \bar{\alpha})$ is an S-saddle point, that is $Y_S = X_S^{\bar{\alpha}}(Y_{\tau_S^*}, \tau_S^*)$ a.s.

In particular, τ_S^* is an optimal stopping time for the agent who wants to minimize her risk measure at time S and $\bar{\alpha}$ corresponds to a worst scenario.

Proof. Since ξ is l.u.s.c. along stopping times, we have

$$Y_S = X_S(Y_{\tau_S^*}, \tau_S^*) = \operatorname*{ess\,inf}_{\alpha} X_S^{\alpha}(Y_{\tau_S^*}, \tau_S^*) = X_S^{\bar{\alpha}}(Y_{\tau_S^*}, \tau_S^*) \quad \text{a.s.}$$

Hence $(\tau_S^*, \bar{\alpha})$ is an S-saddle point. □

This corollary generalizes a result of [12] obtained to the case of Mean-field framework.

We now show the existence of an S-saddle point under weaker assumptions for fixed S in $\mathbb{T}_0$.

Proposition 1.4 (Existence of S-saddle points)**.** *Suppose that for each α in $\mathcal{A}$, $f \leq f^{\alpha}$. Let S in $\mathbb{T}_0$. Suppose that the obstacle ξ is l.u.s.c. along stopping times. Suppose that there exists $\bar{\alpha}$ such that*

$$Y^{\bar{\alpha}}_{\tau_S^*} = \xi_{\tau_S^*} \text{ a.s. } \textit{and } f(t, F(t, Y_t(\cdot)), Y_t, Z_t, k_t) = f^{\hat{\alpha}}(t, F(t, Y_t(\cdot)), Y_t, Z_t, k_t), \;\; S \leq t \leq \tau_S^*, \; dP \otimes dt - \text{a.s.} \tag{VI.10}$$

Then, $(\tau_S^*, \bar{\alpha})$ is an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

Proof. The result follows from the same arguments as above and the sufficient optimality conditions for Mean-field RBSDEs optimization (see Proposition 3.5 2.). □

We now show that there exists a value function under weaker hypotheses.

Theorem 1.2 (Existence and characterization of the common value function - II)**.** *Suppose that for each $\alpha \in \mathcal{A}$, $f \leq f^{\alpha}$. Suppose that for each $\eta > 0$, there exists $\alpha^{\eta} \in \mathcal{A}$ such that*

$$f(t, F(t, Y_t(\cdot)), Y_t, Z_t, k_t) \geq f^{\alpha^{\eta}}(t, F(t, Y_t(\cdot)), Y_t, Z_t, k_t) - \eta, \quad 0 \leq t \leq T,\; dP \otimes dt - a.s. \tag{VI.11}$$

Then, for each $S \in \mathbb{T}_0$, the equality $Y_S = \overline{V}(S) = \underline{V}(S)$ holds a.s.

Proof. By Proposition 3.2, we know that $Y_S = \operatorname{ess\,inf}_{\alpha} Y_S^{\alpha} = \overline{V}(S)$ a.s.
For each $\varepsilon > 0$, by a property of τ_S^{ε} (see Proof of Theorem 2.1,), we have

$$Y_S = X_S(Y_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon}) \text{ a.s.}$$

Now, Assumption (VI.11) holds. Applying an optimization principle for non reflected Mean-field BSDE (Proposition 3.1), we derive that

$$Y_S = X_S(Y_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon}) = \operatorname*{ess\,inf}_{\alpha} X_S^{\alpha}(Y_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon}) \text{ a.s.}$$

The end of the proof is the same as that of Theorem 1.1. □

From the above theorems, the following saddle point criterium clearly follows.

Corollary 1.5 (Saddle point criterium). *Suppose that the assumptions of Theorem 1.1 or Theorem 1.2 are satisfied. Let $S \in \mathbb{T}_0$. For each stopping time $\hat{\tau} \in \mathbb{T}_S$ and for each $\hat{\alpha} \in \mathcal{A}$, the pair $(\hat{\tau}, \hat{\alpha})$ is an S-saddle point if and only if $\hat{\tau}$ is an optimal stopping time for $Y_S = \operatorname{ess\,sup}_{\tau \in \mathbb{T}_S} X_S(\xi_\tau, \tau)$ and $\hat{\alpha}$ is optimal for $Y_S = \operatorname{ess\,inf}_{\alpha \in \mathcal{A}} Y_S^\alpha$.*

Proof. By Theorem 1.1 or 1.2, we have $\underline{V}(S) = \overline{V}(S) = Y_S$ a.s. The result follows from the definition of an S-saddle point (see Definition 1.1). □

Remark 1.6. *Proposition 2.1 gives some necessary conditions for a stopping time $\hat{\tau}$ to be S-optimal, and Proposition 3.4 gives some necessary conditions for a coefficient $\hat{\alpha}$ to be S-optimal. Consequently, under the assumptions of Corollary 1.5, we obtain necessary conditions for a pair $(\hat{\tau}, \hat{\alpha})$ to be an S-saddle point.*

2 Application to the case of multiple priors

We now apply the results of Section 1 to an optimal stopping problem for dynamic risk-measures in the case of multiple priors. Let A be a Polish space (or a Borelian subset of a Polish space) and let $\mathcal{A}$ the set of A-valued predictable processes α. With each coefficient $\alpha \in \mathcal{A}$, is associated a model via a probability measure Q^α called *prior* as well as a dynamic risk measure ρ^α. More precisely, for each $\alpha \in \mathcal{A}$, let Z^α be the solution of the SDE:

$$dZ_t^\alpha = Z_{t^-}^\alpha \left(\beta^1(t, \alpha_t) dW_t + \int_U \beta^2(t, \alpha_t, u) d\tilde{N}(dt, du) \right) ; \quad Z_0^\alpha = 1,$$

where $\beta^1 : (t, \omega, \alpha) \mapsto \beta^1(t, \omega, \alpha)$, is a $\mathcal{P} \otimes \mathcal{B}(A)$-measurable function defined on $[0, T] \times \Omega \times A$ and valued in $[-C, C]$, with $C > 0$, and $\beta^2 : (t, \omega, \alpha, u) \mapsto \beta^2(t, \omega, \alpha, u)$ is a $\mathcal{P} \otimes \mathcal{B}(A) \otimes \mathcal{U}$-measurable function defined on $[0, T] \times \Omega \times A \times U$ which satisfies $dt \otimes dP \otimes d\nu(u)$-a.s.

$$\beta^2(t, \alpha, u) \geq C_1 \quad \text{and} \quad |\beta^2(t, \alpha, u)| \leq \psi(u), \tag{VI.12}$$

with $C_1 > -1$ and ψ is a bounded function in L_ν^p for all $p \geq 1$. Hence, $Z_T^\alpha > 0$ a.s. and, by Proposition A1 in [11], $Z_T^\alpha \in L^p(\mathcal{F}_T)$ for all $p \geq 1$. We suppose that β^1 and β^2 are continuous with respect to α. For each $\alpha \in \mathcal{A}$, let Q^α be the probability measure equivalent to P which admits Z_T^α as density with respect to P on $\mathcal{F}_T$. By Girsanov's theorem, under Q^α, the process $W_t^\alpha := W_t - \int_0^t \beta^1(s, \alpha_s) ds$ is a Brownian motion and N is a Poisson random measure with compensated process $\tilde{N}^\alpha(dt, du) = \tilde{N}(dt, du) - \beta^2(t, \alpha_t, u)\nu(du)dt$ independant from W^α.

For each α, we are going to define a dynamic risk measure induced by a Mean-field BSDE under Q^α and driven by W^α and $\tilde{N}^\alpha$, with a driver defined as follows.

Let $G : [0, T] \times \Omega \times \mathbb{R}^3 \times L_\nu^2 \times A \to \mathbb{R}$; $(t, \omega, x', x, z, \ell, \alpha) \mapsto G(t, \omega, x', x, z, \ell, \alpha)$, be a given $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3) \otimes \mathcal{B}(L_\nu^2) \otimes \mathcal{B}(A)$-measurable function. Suppose G is uniformly Lipschitz with respect to (z, ℓ), continuous with respect to α, and such that $\operatorname{ess\,sup}_{\alpha \in A} |G(\cdot, t, 0, 0, 0, 0, \alpha)| \in \mathbb{H}^{p,T}$, for each $p \geq 2$. Suppose also that

$$G(t, x', x, z, l_1, \alpha) - G(t, x', x, z, l_2, \alpha) \geq \langle \theta_t^{x', x, z, l_1, l_2, \alpha} \, , \, l_1 - l_2 \rangle_\nu, \tag{VI.13}$$

for some adapted process $\theta_t^{x',x,z,l_1,l_2,\alpha}(\cdot)$ satisfying $|\theta_t^{x',x,z,l_1,l_2,\alpha}(u)| \leq \bar{\psi}(u)$, where $\bar{\psi}$ is bounded and in L^p_ν, for all $p \geq 1$, and $\theta_t^{x',x,z,l_1,l_2,\alpha} \geq -1 - C_1$. For each $\alpha \in \mathcal{A}$, the associated driver is given by $G(t, \omega, x', x, z, \ell, \alpha_t(\omega))$. Note that these drivers are equi-Lipschitz.

For each $\alpha \in \mathcal{A}$, let ρ^α be the dynamic risk-measure induced by the Mean-field BSDE associated with $G(., \alpha_t)$ and driven by W^α and $\tilde{N}^\alpha$. More precisely, for each $\tau \in \mathbb{T}_0$ and $\zeta \in L^p(\mathcal{F}_\tau)$ with $p > 2$, there exists a unique solution $(X^\alpha, \pi^\alpha, l^\alpha)$ in $\mathcal{S}^2_\alpha \times \mathbb{H}^2_\alpha \times \mathbb{H}^2_{\alpha,\nu}$ of the Q^α-BSDE

$$- dX_t^\alpha = G(t, \mathbb{E}[\phi(t, X_t^\alpha)], X_t^\alpha, z_t^\alpha, l_t^\alpha, \alpha_t)dt - \pi_t^\alpha dW_t^\alpha - \int_U l_t^\alpha(u)\tilde{N}^\alpha(dt, du); \qquad X_\tau^\alpha = \zeta, \tag{VI.14}$$

driven by W^α and $\tilde{N}^\alpha$, where the expection is taken under the $\mathbf{P}$ measure and ϕ is a Lipschitz function nondecreasing function.

Lemma 2.1. *There exists a unique solution $(X^\alpha, \pi^\alpha, l^\alpha)$ in $\mathcal{S}^2_\alpha \times \mathbb{H}^2_\alpha \times \mathbb{H}^2_{\alpha,\nu}$ of the Q^α-BSDE (VI.14).*

Proof. We define the operator F^α on $L^2_{Q^\alpha}(\mathcal{F}_T)$ such that :

$$F^\alpha(t, X) := \mathbb{E}^{Q^\alpha}\big[(Z_T^\alpha)^{-1}\phi(t, X)\big] = \mathbb{E}\big[\phi(t, X)\big]. \quad \text{for each } X \in L^2_{Q^\alpha}(\mathcal{F}_T)$$

Then we have

$$|F^\alpha(X_1) - F^\alpha(X_2)| \leq C\mathbb{E}^{Q^\alpha}\left[|(Z_T^\alpha)^{-1}(X_1 - X_2)|\right] \leq C\|(Z_T^\alpha)^{-1}\|_{L^2_{Q^\alpha}}\|(X_1 - X_2)\|_{L^2_{Q^\alpha}} \tag{VI.15}$$

where $\|(Z_T^\alpha)^{-1}\|_{L^2_{Q^\alpha}} = (\mathbb{E}\big[(Z_T^\alpha)^{-2}Z_T^\alpha\big])^{\frac{1}{2}} \leq (\|(Z_T^\alpha)^{-1}\|_{L^2_{\mathbf{P}}})^{\frac{1}{4}} < +\infty$. Thus F^α is a Lipschitz mean-field operator on $L^2_{Q^\alpha}(\mathcal{F}_T)$ as defined in (1.3). Then the existence follows by Theorem 1.2. $\square$

The dynamic risk-measure $\rho^\alpha(\zeta, \tau)$ is thus well defined by

$$\rho_t^\alpha(\zeta, \tau) := -X_t^\alpha(\zeta, \tau), \;\; 0 \leq t \leq \tau, \tag{VI.16}$$

with $X^\alpha(\zeta, \tau) = X^\alpha$. Assumption (VI.13) yields the monotonicity property of ρ^α.

We consider an ambiguity averse agent. Her risk measure is given, for each $\tau \in \mathbb{T}_S$ and $\zeta \in L^p(\mathcal{F}_\tau)$, $p > 2$, by

$$\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \rho_S^\alpha(\zeta, \tau) = -\operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} X_S^\alpha(\zeta, \tau) \tag{VI.17}$$

at each stopping time $S \in \mathbb{T}_0$. The financial dynamic position is given here by a RCLL process (ξ_t) which belongs to $\mathcal{S}^p$, for some $p > 2$. At fixed time $S \in \mathbb{T}_0$, the agent wants to choose a stopping time in $\mathbb{T}_S$ so that it minimizes (VI.17), which leads to the mixed control/optimal stopping problem:

$$u(S) := \operatorname*{ess\,inf}_{\tau \in \mathbb{T}_S} \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \rho_S^\alpha(\xi_\tau, \tau) = -\operatorname*{ess\,sup}_{\tau \in \mathbb{T}_S} \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} X_S^\alpha(\xi_\tau, \tau), \tag{VI.18}$$

which corresponds to that studied in Section 1.

Theorem 2.1. *Let (Y, Z, k) be the solution of the mean-field RBSDE associated with obstacle (ξ_t) and Lispchitz driver f, defined for each $(t, \omega, x', x, z, \ell)$ by*

$$f(t, \omega, x', x, z, \ell) := \inf_{\alpha \in A} \{G(t, \omega, x', x, z, \ell, \alpha) + \beta^1(t, \omega, \alpha)\pi + \langle \beta^2(t, \omega, \alpha), \ell \rangle_\nu\}. \qquad \text{(VI.19)}$$

For each $S \in \mathbb{T}_0$, we have

$$Y_S = \overline{V}(S) = \underline{V}(S) \text{ a.s.}$$

In particular, $u(S) = -Y_S$ a.s.

Proof. In order to prove this result, we express the problem in terms of Mean-field BSDEs and Mean-field RBSDEs under probability P and then apply Theorem 1.2.

Fix $\tau \in \mathbb{T}_0$ and $\zeta \in L^p(\mathcal{F}_\tau)$ with $p > 2$. Since $(X^\alpha, \pi^\alpha, l^\alpha)$ is the solution of BSDE (VI.14), it clearly satisfies the following P-BSDE driven by W and $\tilde{N}$

$$-dX_t^\alpha = f^\alpha(t, \mathbb{E}[\phi(t, X_t^\alpha)], X_t^\alpha, Z_t^\alpha, l_t^\alpha)dt - Z_t^\alpha dW_t - \int_U l_t^\alpha(u)\tilde{N}(dt, du); \qquad X_\tau^\alpha = \zeta, \quad \text{(VI.20)}$$

where the driver is given by

$$f^\alpha(t, x', x, z, \ell) := G(t, x', x, z, \ell, \alpha_t) + \beta^1(t, \alpha_t)\pi + \langle \beta^2(t, \alpha_t), \ell \rangle_\nu. \qquad \text{(VI.21)}$$

The process $(X^\alpha, \pi^\alpha, l^\alpha)$ is the solution of P-BSDE (VI.20) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2$ (see proof of Th. 5.9 in [11]). Moreover, for each α, f^α satisfies Assumption 3.1, and f, defined by (VI.19), is a Lipschitz driver (see [11]). By the definition of f and f^α (see (VI.21)), we get that for each $\alpha \in \mathcal{A}$, $f \leq f^\alpha$. For each $\eta > 0$ and each $(t, \omega, x', x, z, l) \in \Omega \times [0, T] \times \mathbb{R}^3 \times L_\nu^2$, there exists $\alpha^\eta \in A$ such that

$$f(t, \omega, x', x, z, \ell) + \eta \geq G(t, \omega, x', x, z, \ell, \alpha^\eta) + \beta^1(t, \omega, \alpha^\eta)\pi + \langle \beta^2(t, \omega, \alpha^\eta), \ell \rangle_\nu.$$

By the section theorem of [2], for each $\eta > 0$, there exists an A-valued predictable process (α_t^η) such that $f(t, \mathbb{E}[\phi(t, Y_t)], Y_t, Z_t, k_t) + \eta \geq f^{\alpha^\eta}(t, \mathbb{E}[\phi(t, Y_t)], Y_t, Z_t, k_t)$. By Theorem 1.2, result follows. $\square$

Chapter VII

Another type of Mean-field BSDEs and related results

Now we introduce the second type of mean-field BSDEs. We give the similar results in parallel to the first type of mean-field BSDEs stated in previous sections. And the proofs are in Appendix.

1 Mean-field BSDEs with jumps

Mean-field BSDEs Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) = (\Omega\times\Omega, \mathcal{F}\times\mathcal{F}, P\times P)$ be the (Non-completed) product of $(\Omega, \mathcal{F}, P)$ with itself. We endow this product space with the filtration $\bar{\mathbb{F}} = \{\mathcal{F}\times\mathcal{F}_t, 0 \leq t \leq T\}$. For any $\theta \in L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ the variable $\theta(.,\omega) : \omega \to \mathbb{R}$ belongs to $L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, $P(d\omega)$-a.s.; we denote its expectation by

$$E'[\theta(.,\omega)] = \int_\Omega \theta(\omega',\omega)P(d\omega').$$

Notice that $E'[\theta] = E'[\theta(.,\omega)] \in L^1(\Omega, \mathcal{F}, P)$ and

$$\bar{E}[\theta] = \int_{\bar{\Omega}} \theta d\bar{p} = \int_\Omega E'[\theta(.,\omega)]P(d\omega) = E[E'[\theta]].$$

The driver of our mean-field BSDE is a function $f = f(\omega',\omega,t,y',z',l',y,z,l) : \bar{\Omega}\times[0,T]\times\mathbb{R}^2\times L^2_\nu\times\mathbb{R}^2\times L^2_\nu \to \mathbb{R}$ which is $\bar{F}$ progressively measurable, for all (y',z',l',y,z,l), and satisfies the following assumptions:

Definition 1.1 (Driver, Lipschitz driver)**.** *A function f is said to be a* driver *if*

- $f : \bar{\Omega}\times[0,T]\times\mathbb{R}^2\times L^2_\nu\times\mathbb{R}^2\times L^2_\nu \to \mathbb{R}$
  $(\omega',\omega,t,y',z',l'(\cdot),y,z,l(\cdot)) \mapsto f(\omega',\omega,t,y',z',l'(\cdot),y,z,l(\cdot))$ *is* $\bar{F}\otimes\mathcal{B}(\mathbb{R}^2)\otimes\mathcal{B}(L^2_\nu)\otimes\mathcal{B}(\mathbb{R}^2)\otimes\mathcal{B}(L^2_\nu)-$ *measurable,*
- $f(.,0,0,0,0,0,0) \in \mathbb{H}^2$.

A driver f is called a Lipschitz driver *if moreover there exists a constant $C \geq 0$ such that $dP\otimes dt$-a.s. , for each $(y'_1,z'_1,l'_1,y_1,z_1,l_1)$, $(y'_2,z'_2,l'_2,y_2,z_2,l_2)$,*

$$|f(\omega,t,y'_1,z'_1,l'_1,y_1,z_1,l_1) - f(\omega,y'_2,z'_2,l'_2,t,y_2,z_2,l_2)| \tag{VII.1}$$
$$\leq C(|y'_1-y'_2| + |z'_1-z'_2| + \|l'_1-l'_2\|_\nu + |y_1-y_2| + |z_1-z_2| + \|l_1-l_2\|_\nu). \tag{VII.2}$$

Let us now recall the definition of a mean-field BSDE with jumps ([8] section 2.2).

Definition 1.2 (Mean field BSDE with jumps [8]). *A solution of a Mean-field BSDE with jumps with terminal time T, terminal condition ξ and driver f consists of a triple of processes $(X, Z, l) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ satisfying*

$$-dX_t = \mathbb{E}'\left[f(t, \omega, X'_t, Z'_t, l'_t(\cdot), X_t, Z_t, l_t(\cdot))\right] dt - Z_t dW_t - \int_{\mathbf{E}} l_t(e)\tilde{N}(dt, de); \qquad \text{(VII.3)}$$
$$X_T = \xi.$$

where X is a RCLL optional process, and Z (resp. l) is an $\mathbb{R}$-valued predictable process defined on $\Omega \times [0, T]$ (resp. $\Omega \times [0, T] \times \mathbb{R}^*$) such that the stochastic integral with respect to W (resp. $\tilde{N}$) is well defined. We denote by $(X(\xi, T), Z(\xi, T), l_t(\xi, T))$ the solution of the Mean-field BSDE associated with terminal time T and (ξ, f).

Then, recall that for each Lipschitz driver f, and each terminal condition $\xi \in L^2(\mathcal{F}_T)$, there exists a unique solution $(X, Z, l) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ satisfying (VII.3).

Comparison Results

Assumption 3.2. *For each $(\hat{x}, \hat{z}, \hat{l}) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$, we define for all $(x, z, \ell) \in \mathbb{R}^2 \times L^2_\nu$*

$$f^{\hat{x},\hat{z},\hat{\ell}}(t, x, z, \ell) = \mathbb{E}'[f(\cdot, \omega, t, \hat{x}'_t, \hat{z}'_t, \hat{\ell}'_t(\cdot), x, z, \ell(\cdot)]. \ \ dP \otimes dt - a.s$$

Assume that $dP \otimes dt$-a.s for each $(x, z, \ell_1, \ell_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$f^{\hat{x},\hat{z},\hat{\ell}}(t, x, z, \ell_1) - f^{\hat{x},\hat{z},\hat{\ell}}(t, x, z, \ell_2) \geq \langle \theta_t^{\hat{x},\hat{z},\hat{\ell}}(x, z, \ell_1, \ell_2)\,,\, \ell_1 - \ell_2 \rangle_\nu,$$

with

$$\theta_t^{\hat{x},\hat{z},\hat{\ell}} : [0, T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \to L^2_\nu\, ;\ (\omega, t, x, z, \ell_1, \ell_2) \mapsto \theta_t^{\hat{x},\hat{z},\hat{\ell}}(x, z, \ell_1, \ell_2)(\omega, .)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable, bounded, and satisfying $dP \otimes dt \otimes d\nu(u)$-a.s., for each $(x, z, \ell_1, \ell_2) \in \mathbb{R}^2 \times L^2_\nu \times \mathbb{R}^2 \times (L^2_\nu)^2$,

$$\theta_t^{\hat{x},\hat{z},\hat{\ell}}(x, z, \ell_1, \ell_2)(u) \geq -1 \quad \text{and} \quad |\theta_t^{\hat{x},\hat{z},\hat{\ell}}(x, z, \ell_1, \ell_2)(u)| \leq \psi(u), \qquad \text{(VII.4)}$$

where $\psi \in L^2_\nu$.

Theorem 1.1 (Comparison Theorem for Mean-field BSDEs with jumps). *Let $f_i = f_i(\omega', \omega, t, x', z', l', x, z, l)$, $i = 1, 2$, be two Lipschitz drivers, and one of them satisfy Assumption 3.2. Furthermore, we assume:*

- *One of the both coefficients is independent of z';*
- *One of the both coefficients is independent of l';*
- *One of the both coefficients is nondecreasing in x';*

Let $\xi_1, \xi_2 \in L^2(\mathcal{F}_T)$ and denote by (X^1, Z^1, l^1) and (X^2, Z^2, l^2) the solution of the mean-field BSDE with jump (VII.3) associated with (ξ_1, f_1) and (ξ_2, f_2). Then suppose that

- $\xi_1 \geq \xi_2$, *a.s.*
- $f_1(\omega', \omega, t, y', z', l'(\cdot), y, z, l(\cdot)) \geq f_2(\omega', \omega, t, y', z', l'(\cdot), y, z, l(\cdot))$, $dt \otimes dP$ *a.s.* *for all* $(y', z', l'(\cdot), y, z, l(\cdot)) \in \mathbb{R}^2 \times L^2_\nu \times \mathbb{R}^2 \times L^2_\nu$

Then we have $X^1_t \geq X^2_t$, $\forall t \in [0,T]$ *a.s.*

In the sequel, we give comparison theorem for mean-field reflected BSDEs with jumps

Theorem 1.2 (Strict comparison for Mean-field BSDEs with jumps)**.** *Suppose the assumption of Theorem 1.1 holds. Moreover we assume one of the coefficient satisfy Assumption 3.2 with strict inequality* $\theta^{\hat{x},\hat{z},\hat{\ell}}_t(x, z, \ell_1, \ell_2)(u) > -1$ $dt \otimes dP$*- a.s. If* $X^1_{t_0} = X^2_{t_0}$ *a.s. for some* $t_0 \in [0,T]$*, then* $X^1_. = X^2_.$ *a.s. on* $[t_0, T]$.

Remark 1.3. *We can relax the assumption in the Theorem 1.2 by assuming the condition* $\theta^{\hat{x},\hat{z},\hat{\ell}}_t(x, z, \ell_1, \ell_2)(u) > -1$ $dt \otimes dP$*- a.s. hold only along the solutions. i.e.* $\theta^{X^1,\cdot,\cdot}_t(X^2_t, Z^2_t, l^1_t, l^2_t)(u) > -1$ $dt \otimes dP$*- a.s.*

2 Optimal stopping for Mean-Field BSDEs with jumps

Reflected mean-field BSDEs with RCLL obstacle

Definition 2.1. *([12] Definition 2.4) A process* $(Y, Z, k(.), A)$ *is said to be a solution of the mean-field reflected BSDE with jump associated with driver* f *and obstacle* $\xi_.$ *if*

$$(Y, Z, k(.), A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$$

$$-dY_t = \mathbb{E}'\left[f(t, \omega, X'_t, Z'_t, k'_t(\cdot), X_t, Z_t, k_t(\cdot))\right]dt + dA_t - Z_t dW_t - \int_U k_t(u)\tilde{N}(dt, du); \quad Y_T = \xi_T, \tag{VII.5}$$

$Y_t \geq \xi_t,\ 0 \leq t \leq T$ *a.s.*,
A is a nondecreasing RCLL continous process with $A_0 = 0$ *and such that*

$$\int_0^T (Y_t - \xi_t) dA^c_t = 0 \text{ a.s. and } \Delta A^d_t = -\Delta Y_t \mathbf{1}_{\{Y_{t-} = \xi_{t-}\}} \quad \text{a.s.}$$

For $\beta > 0$ and $\phi \in \mathbb{H}^{2,T}$, we introduce the norm $\|\phi\|^2_{\beta,T} := E[\int_0^T e^{\beta s}\phi^2_s ds]$ and for $l \in \mathbb{H}^{2,T}_\nu$, we set $\|l\|^2_{\nu,\beta,T} := E[\int_0^T e^{\beta s}\|l_s\|^2_\nu ds]$.

We now show an existence and uniqueness result for mean-field reflected BSDEs with jumps, in the general case of RCLL obstacle.

Theorem 2.1. *(Existence and Uniqueness)*
Let $\xi_.$ *be a RCLL adapted process in* $\mathcal{S}^2$ *and let* f *be a Lipschitz driver. The mean-field RBSDE (VII.5) admits a unique solution* $(Y, Z, k(.), A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$. *If* (ξ_t) *is left-upper semicontinuous (l.u.s.c.) over stopping times, then* (A_t) *is continuous.*

Characterization of the solution of Optimal stopping for mean-field BSDEs as the solution of mean-field reflected BSDE

Theorem 2.2. *Let $T > 0$ be the terminal time. Let $(\xi_t, 0 \le t \le T)$ be an RCLL process in $\mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption 3.2. Furthermore, we assume the assumption in Theorem 1.1 are also satisfied. Suppose now that $(Y, Z, k(\cdot), A)$ is the solution of the mean-field reflected BSDE (VII.5). Then*

- *For each stopping time $S \in \mathbb{T}_0$, we have*

$$Y_S = \operatorname*{ess\,sup}_{\tau\in\mathbb{T}_S} X_S(\xi_\tau, \tau) \quad \text{a.s.} \tag{VII.6}$$

where for $\tau \in \mathbb{T}_S$, $X_\cdot(\xi_\tau, \tau)$ is the solution of the mean-field BSDE (VII.3) associated with terminal time τ, terminal condition ξ_τ, and driver f.

- *For each $S \in \mathbb{T}_0$ and each $\varepsilon > 0$, let τ_S^ε be the stopping time defined by*

$$\tau_S^\varepsilon = \inf\{t \ge S,\ Y_t \le \xi_t + \varepsilon\}. \tag{VII.7}$$

We have

$$Y_S \ \le\ X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) + K\varepsilon \quad \text{a.s.}, \tag{VII.8}$$

*where $K = K(T, C)$ is a constant which only depends on T and the Lipschitz constant C of f. In other words, τ_S^ε is a $(K\varepsilon)$-*optimal stopping time *for* (VII.6).

Optimal Stopping times By the strict comparison theorem for Mean-field BSDEs (Theorem 1.2), we derive the following optimality criterium for the optimal stopping time problem (VII.6).

In this subsection and next section, when we assume Assumption 3.2 and the assumption in Theorem 1.1, we choose the situation as in the Proof of Theorem 1.2. That is, we choose f_1 is independent of both z' and l', f_2 is nondecreasing in x'. And Assumption 3.2 with strict inequality holds for f_1. The other symmetric case can be showed similarly.

Proposition 2.2 (Optimality criterium.)**.** *Let $(\xi_t, 0 \le t \le T)$ be a RCLL process in $\mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption 3.2 and the assumption in Theorem 1.1. Let $S \in \mathbb{T}_0$ and let $\hat\tau \in \mathbb{T}_S$. Suppose that in Assumption 3.2, we have*

$$\theta_t(Y_t, Z_t, k_t, l_t^{\hat\tau}) > -1, \quad dt \otimes dP - \text{a.s.} \tag{VII.9}$$

where $(X^{\hat\tau}, Z^{\hat\tau}, l^{\hat\tau}) = (X(\xi_{\hat\tau}, \hat\tau), Z(\xi_{\hat\tau}, \hat\tau), l(\xi_{\hat\tau}, \hat\tau))$ is the solution of the Mean-field BSDE associated with terminal conditions $(\hat\tau,\ \xi_{\hat\tau})$.
The stopping time $\hat\tau$ is S-optimal, i.e.

$$Y_S = \operatorname*{ess\,sup}_{\tau\in\mathbb{T}_S} X_S(\xi_\tau, \tau) = X_S(\xi_{\hat\tau}, \hat\tau) \quad a.s. \tag{VII.10}$$

if and only if

$$Y_s = X_s(\xi_{\hat\tau}, \hat\tau), \quad S \le s \le \hat\tau \quad \text{a.s.} \tag{VII.11}$$

that is if and only if $(Y_s,\ S \le s \le \hat\tau)$ is the solution of the non reflected mean-field BSDE associated with terminal time $\hat\tau$ and terminal condition $\xi_{\hat\tau}$.

Again, under a left regularity condition on the obstacle, when τ_S^ε tends to an S- optimal stopping time as ε tends to 0, we have the following results.

Theorem 2.3. *Let $(\xi_t, 0 \leq t \leq T)$ be a RCLL process in $\mathcal{S}^2$, assumed to be l.u.s.c. along stopping times, and let f be a Lipschitz driver satisfying Assumption 3.2. Let $S \in \mathbb{T}_0$.*

(i) The stopping time $\tilde{\tau}_S$ defined by

$$\tilde{\tau}_S := \lim_{\varepsilon \downarrow 0} \uparrow \tau_S^{\varepsilon},$$

with τ_S^{ε} given in (VII.7)*, is an S-optimal stopping time.*

(ii) The stopping time τ_S^* defined by

$$\tau_S^* := \inf\{u \geq S;\, Y_u = \xi_u\}$$

is an S-optimal stopping time and we have

$$Y_s = X_s(\xi_{\tau_S^*}, \tau_S^*), \quad S \leq s \leq \tau_S^* \quad \text{a.s.}$$

We also have $\tau_S^* \geq \tilde{\tau}_S$ a.s.

(iii) The stopping time $\bar{\tau}_S$ defined by

$$\bar{\tau}_S := \inf\{u \geq S;\, A_u - A_S > 0\}$$

is an S-optimal stopping time.

(iv) Suppose moreover that in Assumption 3.2, for all x, π, l_1, l_2, we have

$$\theta_t^{\hat{x},\hat{z},\hat{\ell}}(x, z, l^1, l^2)(u) > -1 \quad dt \otimes dP - \text{a.s.} \tag{VII.12}$$

Then, $\tau_S^* = \tilde{\tau}_S$ a.s. Moreover τ_S^* is the minimal and $\bar{\tau}_S$ is the maximal S-optimal stopping time.

Comparison theorems for mean-field RBSDEs and optimization problems

Theorem 2.4 (Comparison)**.** *Let ξ^1, ξ^2 be two RCLL obstacle processes in $\mathcal{S}^2$. Let f^1 and f^2 be Lipschitz drivers satisfying Assumption 3.2 and the assumption in Theorem 1.1. Suppose that*

- *$\xi_t^2 \leq \xi_t^1$, $0 \leq t \leq T$ a.s.*
- *$f^1(\omega', \omega, t, y', z', l'(\cdot), y, z, l(\cdot)) \geq f^2(\omega', \omega, t, y', z', l'(\cdot), y, z, l(\cdot))$ $dP \otimes dt$,a.s. for all $(y', z', l'(\cdot), y, z, l(\cdot)) \in \mathbb{R}^2 \times L_\nu^2 \times \mathbb{R}^2 \times L_\nu^2$*

Let (Y^i, Z^i, k^i, A^i) be the solution of the mean-field RBSDE associated with (ξ^i, f^i) , $i = 1, 2$. Then,

$$Y_t^2 \leq Y_t^1,\ \forall t \in [0, T] \text{ a.s.}$$

Theorem 2.5 (Strict comparison)**.** *Suppose that the assumptions of the comparison theorem (Theorem 2.4) hold and that the driver f^1 satisfies Assumption 3.2 with*

$$\theta_t^{\hat{x},\hat{z},\hat{\ell}}(x, z, l^1, l^2)(u) > -1 \quad dt \otimes dP - \text{a.s.} \tag{VII.13}$$

Let S in $\mathbb{T}_0$ and suppose that $Y_S^1 = Y_S^2$ a.s.

(i) *Suppose that* ξ^1 *and* ξ^2 *are l.u.s.c. along stopping times.* *Let* $\tau_i^* = \tau_{i,S}^* := \inf\{s \geq S;\, Y_s^i = \xi_s^i\},\ i = 1,2.$ *Then,*

$$Y_t^1 = Y_t^2,\ \ S \leq t \leq \tau_1^* \wedge \tau_2^* \ \ a.s, \quad and$$

$$\mathbb{E}'\left[f^1(t, Y_t^{1'}, Y_t^2, Z_t^2, k_t^2)\right] = \mathbb{E}'\left[f^2(t, Y_t^{2'}, Z_t^{2'}, k_t^{2'}, Y_t^2, Z_t^2, k_t^2)\right],\ \ S \leq t \leq \tau_1^* \wedge \tau_2^*,\ dP\otimes dt-a.s. \tag{VII.14}$$

Moreover if $\xi^1 = \xi^2$ *a.s., then* $\tau_1^* = \tau_2^*$ *a.s. and* $Y^1_{\tau_1^*} = Y^2_{\tau_1^*} = \xi^1_{\tau_1^*}$ *a.s*

(ii) *Consider the general case when* ξ^1 *and* ξ^2 *are not supposed to be l.u.s.c. along stopping times. For* $\varepsilon > 0$*, define*

$$\tau_i^\varepsilon := \inf\{t \geq S,\ Y_t^i \leq \xi_t^i + \varepsilon\} \quad and \quad \tilde{\tau}_i := \lim_{\varepsilon \downarrow 0} \uparrow \tau_i^\varepsilon \ \ i = 1,2.$$

Then, $Y_t^1 = Y_t^2,\ \ S \leq t < \tilde{\tau}_1 \wedge \tilde{\tau}_2$ *a.s. Moreover,*

$$\mathbb{E}'\left[f^1(t, Y_t^{1'}, Y_t^2, Z_t^2, k_t^2)\right] = \mathbb{E}'\left[f^2(t, Y_t^{2'}, Z_t^{2'}, k_t^{2'}, Y_t^2, Z_t^2, k_t^2)\right],\ \ S \leq t \leq \tilde{\tau}_1 \wedge \tilde{\tau}_2,\ dP\otimes dt-a.s.$$

and if $\xi^1 = \xi^2$ *a.s., then for each* $\varepsilon > 0$, $\tau_1^\varepsilon = \tau_2^\varepsilon$ *a.s. and* $\tilde{\tau}_1 = \tilde{\tau}_2$.

Optimization problems for RBSDEs Let ξ in $\mathcal{S}^2$ and let $(f, f^\alpha; \alpha \in \mathcal{A})$ be a family of Lipschitz drivers satisfying Assumption 3.2. In Assumption 3.2, the coefficient associated with f^α (resp. f), is denoted by $\theta_t^{\alpha,\hat{x},\hat{z},\hat{\ell},x,z,l}$ (resp. $\theta_t^{\hat{x},\hat{z},\hat{\ell},x,z,l}$). And from this section on, we furthermore assume the assumption in Theorem 1.1 are satisfied. We denote by (Y, Z, k) the solution of the Mean-field RBSDE associated to obstacle (ξ_t) and driver f, and by $(Y^\alpha, Z^\alpha, k^\alpha)$ the solution of the Mean-field RBSDE associated with obstacle (ξ_t) and driver f^α.

For each $\tau \in \mathbb{T}_0$ and $\zeta \in L^2(\mathcal{F}_\tau)$, we denote by $(X(\zeta,\tau), \pi(\zeta,\tau), l(\zeta,\tau))$ the solution of the Mean-field BSDE associated with driver f, terminal conditions ζ, τ, and by $(X^\alpha(\zeta,\tau), \pi^\alpha(\zeta,\tau), l^\alpha(\zeta,\tau))$ the solution of the Mean-field BSDE associated with driver f^α and terminal conditions ζ, τ.

Let $S \in \mathbb{T}_0$. We consider the following optimization problem

$$\operatorname*{ess\,inf}_\alpha Y_S^\alpha. \tag{VII.15}$$

Proposition 2.3 (Optimization principle for Mean-field BSDEs I)**.** *Suppose that*

(i) *For each* $\alpha \in \mathcal{A}$, $f(t, y', z', k', y, z, k) \leq f^\alpha(t, y', z', k', y, z, k)$, *for all* $(y', z', k', y, z, k) \in \mathbb{R}^2 \times \mathcal{L}_\nu^2 \times \mathbb{R}^2 \times \mathcal{L}_\nu^2$; $dt \otimes dP - a.s.$

(ii) *There exists* $\bar{\alpha} \in \mathcal{A}$ *such that*

$$\mathbb{E}' f(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t) = \mathbb{E}' f^{\bar{\alpha}}(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t),\ \ 0 \leq t \leq T,\ \ dt \otimes dP - \text{a.s.} \tag{VII.16}$$

Then, for each $S \in \mathbb{T}_0$,

$$Y_S = \operatorname*{ess\,inf}_\alpha Y_S^\alpha = Y_S^{\bar{\alpha}} \quad \text{a.s.} \tag{VII.17}$$

Proposition 2.4 (Optimization principle for Mean-field RBSDEs I)**.** *Suppose that*

(i) *For each $\alpha \in \mathcal{A}$, $f(t,y',z',k',y,z,k) \leq f^\alpha(t,y',z',k',y,z,k)$, for all $(y',z',k',y,z,k) \in \mathbb{R}^2 \times \mathcal{L}^2_\nu \times \mathbb{R}^2 \times \mathcal{L}^2_\nu$; $dt \otimes dP - a.s.$*

(ii) *There exists $\bar{\alpha} \in \mathcal{A}$ such that*

$$\mathbb{E}' f(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t) = \mathbb{E}' f^{\bar{\alpha}}(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t), \quad 0 \leq t \leq T, \quad dt \otimes dP - \text{a.s.} \tag{VII.18}$$

Then, for each $S \in \mathbb{T}_0$,

$$Y_S = \operatorname{ess\,inf}_{\alpha} Y_S^\alpha = Y_S^{\bar{\alpha}} \quad \text{a.s.} \tag{VII.19}$$

Proof. We apply the same argument as in the Proof of Proposition 3.1 with only the replacement of the comparison theorem for Mean-field RBSDEs yields (see Theorem 2.4).

□

Using an estimate on mean-field RBSDEs (see (VII.42)), we derive a similar characterization of the value function of the problem (VII.15) under weaker hypotheses.

Proposition 2.5 (Optimization principle for RBSDEs II)**.** *Suppose that the drivers f^α, $\alpha \in \mathcal{A}$ satisfy $f \leq f^\alpha$ and are equi-Lipschitz with constant C.*
Suppose moreover that for each $\eta > 0$, there exists $\alpha^\eta \in \mathcal{A}$ such that

$$\mathbb{E}' f(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t) \geq \mathbb{E}' f^{\alpha^\eta}(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t) - \eta, \quad 0 \leq t \leq T, \, dP \otimes dt - \text{a.s.} \tag{VII.20}$$

Then, for each $S \in \mathbb{T}_0$, we have

$$Y_S = \operatorname{ess\,inf}_{\alpha} Y_S^\alpha \quad \text{a.s.} \tag{VII.21}$$

Proof. Since $f \leq f^\alpha$, we have $Y \leq Y^\alpha$ a.s. for each $\alpha \in \mathcal{A}$. It follows that for each $S \in \mathbb{T}_0$, we have $Y_S \leq \operatorname{ess\,inf}_\alpha Y_S^\alpha$ a.s. Since Assumption (VII.20) holds, by using estimate (VII.42), with $\eta = \frac{1}{C^2}$ and $\beta = 5C^2 + 2C$, we derive that there exists a constant $K \geq 0$, which depends only on C and T, such that, for each $\eta > 0$ and for each $S \in \mathbb{T}_0$,

$$Y_S + K\,\eta \geq Y_S^{\alpha^\eta} \geq \operatorname{ess\,inf}_{\alpha} Y_S^\alpha \quad \text{a.s.}$$

Equality (VII.21) thus follows. □

By using the strict comparison theorem for reflected Mean-field BSDEs (see Theorem 2.5), we provide now some necessary optimality conditions at a given time $S \in \mathbb{T}_0$.

Proposition 2.6 (Necessary optimality conditions)**.** *Suppose that the assumptions of Proposition 2.4 or Proposition 2.5 hold. Let $\hat{\alpha} \in \mathcal{A}$, and suppose that in Assumption 3.2 the coefficient $\theta^{\hat{\alpha}}$ corresponding to driver $f^{\hat{\alpha}}$ satisfies $\theta_t^{\hat{\alpha},\hat{x},\hat{z},\hat{\ell},x,z,l^1,l^2}(u) > -1$, for each x, π, ℓ_1, ℓ_2. Let $S \in \mathbb{T}_0$. Suppose that $\hat{\alpha}$ is S-optimal, i.e.*

$$ess \inf_{\alpha} Y_S^\alpha = Y_S^{\hat{\alpha}} \quad \text{a.s.} \tag{VII.22}$$

(i) *Suppose ξ is l.u.s.c. along stopping times. Let $\tau_S^* := \inf\{t \geq S,\ Y_t = \xi_t\}$. Then*

$$Y^{\hat{\alpha}}_{\tau_S^*} = \xi_{\tau_S^*} \text{ a.s.}\,;\mathbb{E}'f(t,Y_t^{'},Z_t^{'},k_t^{'},Y_t,Z_t,k_t) = \mathbb{E}'f^{\hat{\alpha}}(t,Y_t^{'},Z_t^{'},k_t^{'},Y_t,Z_t,k_t),\ S \leq t \leq \tau_S^*,\ dP\otimes dt-a.s. \tag{VII.23}$$

(ii) *Consider the case when ξ is not supposed to be l.u.s.c. along stopping times. For each $\varepsilon > 0$, let $\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\}$. Then for each $\varepsilon > 0$,*

$$Y^{\hat{\alpha}}_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon \text{ a.s.}\,;\mathbb{E}'f(t,Y_t^{'},Z_t^{'},k_t^{'},Y_t,Z_t,k_t) = \mathbb{E}'f^{\hat{\alpha}}(t,Y_t^{'},Z_t^{'},k_t^{'},Y_t,Z_t,k_t),\ S \leq t \leq \tau_S^\varepsilon,\ dP\otimes dt-\text{a.s.} \tag{VII.24}$$

Proposition 2.7 (Sufficient optimality conditions)**.**
Suppose that for each $\alpha \in \mathcal{A}$, $f \leq f^\alpha$. Let $\hat{\alpha} \in \mathcal{A}$ and $S \in \mathbb{T}_0$.

(i) *Suppose ξ is l.u.s.c. along stopping times.*
If equalities (VII.23) *hold, then $\hat{\alpha}$ is S-optimal, that is, $ess\inf_\alpha Y_S^\alpha = Y_S^{\hat{\alpha}}$ a.s.*

(ii) *Consider the case when ξ is not supposed to be l.u.s.c. along stopping times.*
If for each $\varepsilon > 0$, conditions (VII.24) *hold, then $\hat{\alpha}$ is S-optimal.*

In both cases, we get $Y_S = ess\inf_\alpha Y_S^\alpha$ a.s.

3 Robust optimal stopping problem

We now consider the optimal stopping problem when there is ambiguity on the risk-measure modeling. Let $\{f^\alpha, \alpha \in \mathcal{A}\}$ be a given family of Lipschitz drivers satisfying Assumption (3.2). For each $\alpha \in \mathcal{A}$, let ρ^α be the risk measure induced by the Mean-field BSDE with driver f^α, defined as follows: for each terminal time $\tau \in \mathbb{T}_0$ and position $\zeta \in L^2(\mathcal{F}_\tau)$, set

$$\rho_t^\alpha(\zeta,\tau) := -X_t^\alpha(\zeta,\tau), \quad 0 \leq t \leq T,$$

where $X_t^\alpha(\zeta,\tau)$ denotes the solution of the Mean-field BSDE associated with driver f^α, terminal condition ζ and terminal time τ. We consider an agent who is averse to ambiguity, and we define her risk measure of position ζ, at each time S in $\mathbb{T}_0$ with $S \leq \tau$ a.s., as the supremum over α of the associated risk-measures $\rho_S^\alpha(\zeta,\tau)$ that is,

$$\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \rho_S^\alpha(\zeta,\tau) = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} -X_S^\alpha(\zeta,\tau).$$

Let (ξ_t) be a dynamic position, given by an RCLL adapted process (ξ_t) in $\mathcal{S}^2$. At time $S \in \mathbb{T}_0$, the agent wants to find a stopping time $\tau \in \mathbb{T}_S$ which minimizes her risk measure. At time S, her value function is defined as

$$u(S) := \operatorname*{ess\,inf}_{\tau\in\mathbb{T}_S} \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \rho_S^\alpha(\xi_\tau,\tau). \tag{VII.25}$$

Let $S \in \mathbb{T}_0$. Define the *first value function at time S* as

$$\overline{V}(S) := \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}} \operatorname*{ess\,sup}_{\tau\in\mathbb{T}_S} X_S^\alpha(\xi_\tau,\tau), \tag{VII.26}$$

and the *second value function at time S* as

$$\underline{V}(S) := \text{ess}\sup_{\tau \in \mathbb{T}_S} \text{ess}\inf_{\alpha \in \mathcal{A}} X_S^\alpha(\xi_\tau, \tau). \tag{VII.27}$$

Note that $\underline{V}(S) = -u(S)$ a.s.
By definition, we say that there exists a *value function* at time S for the game problem if $\underline{V}(S) = \overline{V}(S)$ a.s.

Definition 3.1 (S-Saddle point). *Let $S \in \mathbb{T}_0$. A pair $(\hat{\tau}, \hat{\alpha}) \in \mathbb{T}_S \times \mathcal{A}$ is called a S-*saddle point *if*

- $\underline{V}(S) = \overline{V}(S)$ *a.s.*,
- *the essential infimum in (VII.26) is attained at $\hat{\alpha}$,*
- *the essential supremum in (VII.27) is attained at $\hat{\tau}$.*

By classical results, for each $S \in \mathbb{T}_0$, $(\hat{\tau}, \hat{\alpha})$ is a S-saddle point if and only if for each $(\tau, \alpha) \in \mathbb{T}_S \times \mathcal{A}$,

$$X_S^{\hat{\alpha}}(\xi_\tau, \tau) \leq \; X_S^{\hat{\alpha}}(\xi_{\hat{\tau}}, \hat{\tau}) \leq \; X_S^{\alpha}(\xi_{\hat{\tau}}, \hat{\tau}) \text{ a.s.} \tag{VII.28}$$

Note that for each $S \in \mathbb{T}_0$, the inequality $\underline{V}(S) \leq \overline{V}(S)$ a.s. clearly holds. We want to determine when the equality holds, characterize the value function, and address the question of existence of a S-saddle point.

Again we can relate the game problem to an optimization problem for mean-field RBSDEs. Let $(Y^\alpha, Z^\alpha, k^\alpha)$ be the solution of the mean-field RBSDE with obstacle (ξ_t) and driver f^α. For each $\tau \in \mathbb{T}_0$ and $\zeta \in L^2(\mathcal{F}_\tau)$, let $X^\alpha(\zeta, \tau)$ be the solution of the mean-field BSDE with driver f^α and terminal conditions (ζ, τ). By the characterization of mean-field RBSDEs (see Theorem 2.2), for each $S \in \mathbb{T}_0$, we have $Y_S^\alpha = \text{ess}\sup_{\tau \in \mathbb{T}_S} X_S^\alpha(\xi_\tau, \tau)$ a.s. It follows that

$$\overline{V}(S) = \text{ess}\inf_{\alpha \in \mathcal{A}} Y_S^\alpha \quad \text{a .s.} \tag{VII.29}$$

Let f be a Lipschitz driver satisfying Assumption (3.2). Let (Y, Z, k) be the solution of the mean-field RBSDE with obstacle (ξ_t) and driver f. For each $\tau \in \mathbb{T}_0$ and $\zeta \in L^2(\mathcal{F}_\tau)$, let $X(\zeta, \tau)$ be the solution of the mean-field BSDE with driver f and terminal conditions (ζ, τ).

Theorem 3.1 (Existence and characterization of the common value function - I). *Suppose that the drivers f^α, $\alpha \in \mathcal{A}$ satisfy $f \leq f^\alpha$ and are equi-Lipschitz with constant C. Suppose that there exists $\bar{\alpha}$ such that*

$$\mathbb{E}' f(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t) = \mathbb{E}' f^{\hat{\alpha}}(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t), 0 \leq t \leq T, \;\; dt \otimes dP - a.s. \tag{VII.30}$$

Then, there exists a value function, which is characterized as the solution of the RBSDE with obstacle (ξ_t) and driver f, that is, for each $S \in \mathbb{T}_0$, we have

$$Y_S = \overline{V}(S) = \underline{V}(S) \text{ a.s.}$$

In particular, the minimal risk measure, defined by (VII.25)*, satisfies for each $S \in \mathbb{T}_0$*

$$u(S) = -Y_S \text{ a.s.}$$

Proposition 3.2 (Existence of saddle points). *Suppose that the assumptions of Theorem 3.1 are satisfied and that the obstacle ξ is l.u.s.c. along stopping times. For each $S \in \mathbb{T}_0$, let*

$$\tau_S^* := \inf\{u \geq S;\, Y_u = \xi_u\}.$$

Then, $(\tau_S^*, \bar{\alpha})$ is an S-saddle point, that is $Y_S = X_S^{\bar{\alpha}}(Y_{\tau_S^*}, \tau_S^*)$ a.s.

In particular, τ_S^* is an optimal stopping time for the agent who wants to minimize her risk measure at time S and $\bar{\alpha}$ corresponds to a worst scenario.

We now show the existence of an S-saddle point under weaker assumptions for fixed S in $\mathbb{T}_0$.

Proposition 3.3 (Existence of S-saddle points). *Suppose that for each α in $\mathcal{A}$, $f \leq f^{\alpha}$. Let S in $\mathbb{T}_0$. Suppose that the obstacle ξ is l.u.s.c. along stopping times. Suppose that there exists $\bar{\alpha}$ such that*

$$Y^{\bar{\alpha}}_{\tau_S^*} = \xi_{\tau_S^*} \text{ a.s. } \textit{and } \mathbb{E}' f(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t) = \mathbb{E}' f^{\hat{\alpha}}(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t),\;\; S \leq t \leq \tau_S^*,\; dP \otimes dt - \text{a.s.} \tag{VII.31}$$

Then, $(\tau_S^*, \bar{\alpha})$ is an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

We now show that there exists a value function under weaker hypotheses.

Theorem 3.2 (Existence and characterization of the common value function - II). *Suppose that for each $\alpha \in \mathcal{A}$, $f \leq f^{\alpha}$. Suppose that for each $\eta > 0$, there exists $\alpha^{\eta} \in \mathcal{A}$ such that*

$$\mathbb{E}' f(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t) \geq \mathbb{E}' f^{\hat{\alpha}}(t, Y_t^{'}, Z_t^{'}, k_t^{'}, Y_t, Z_t, k_t) - \eta, \quad 0 \leq t \leq T,\, dP \otimes dt - a.s. \tag{VII.32}$$

Then, for each $S \in \mathbb{T}_0$, the equality $Y_S = \overline{V}(S) = \underline{V}(S)$ holds a.s.

From the above theorems, the following saddle point criterium clearly follows.

Proposition 3.4 (Saddle point criterium). *Suppose that the assumptions of Theorem 3.1 or Theorem 3.2 are satisfied. Let $S \in \mathbb{T}_0$. For each stopping time $\hat{\tau} \in \mathbb{T}_S$ and for each $\hat{\alpha} \in \mathcal{A}$, the pair $(\hat{\tau}, \hat{\alpha})$ is an S-saddle point if and only if $\hat{\tau}$ is an optimal stopping time for $Y_S = \operatorname{ess\,sup}_{\tau \in \mathbb{T}_S} X_S(\xi_\tau, \tau)$ and $\hat{\alpha}$ is optimal for $Y_S = \operatorname{ess\,inf}_{\alpha \in \mathcal{A}} Y_S^{\alpha}$.*

4 Appendix

Proofs of Theorem 1.1 Proof. For $i = 1, 2$, let $(X_s^{i,n}, Z_s^{i,n}, l_s^{i,n})$ be the solution of the following iterating BSDE with jumps

$$X_s^{i,n} = \xi_i + \int_t^T \mathbb{E}^{'}[f_i(s, (X_s^{i,n-1})', (Z_s^{i,n-1})', (l_s^{i,n-1})', X_s^{i,n}, Z_s^{i,n}, l_s^{i,n})]ds - \int_t^T Z_s^{i,n} dB_s - \int_t^T \int_{\mathbf{E}} l_t(e)\tilde{N}(dt, de) \tag{VII.33}$$

For $n \geq 1$ and $t \in [0, 1]$. For $n = 0$, we set $(X_s^{i,0}, Z_s^{i,0}, l_s^{i,0}) = (0, 0, 0)$.
Without loss of generality, we assume that f_1 satisfies Assumption 3.2 and is independent of z', while f_2 is independent of l' and is nondecreasing in x'.

Now we define

$$\tilde{f}_1^n(s,x,z,l) = \mathbb{E}^{'}[f_1(s,(X_s^{1,n-1})',(l_s^{1,n-1})',x,z,l)], \tag{VII.34}$$
$$\tilde{f}_2^n(s,x,z,l) = \mathbb{E}^{'}[f_2(s,(X_s^{2,n-1})',(Z_s^{2,n-1})',x,z,l)] \tag{VII.35}$$

Then obviously we have $\tilde{f}_1^1 \le \tilde{f}_2^1$ and $\tilde{f}_1^1$ satisfy the monotone assumption in Theorem 4.2 [11] due to Assumption 3.2. Thus by the classic comparison theorem for BSDEs (Theorem 4.2 [11]), we have

$$X_s^{1,1} \le X_s^{2,1}, \quad s \in [0,T]. \tag{VII.36}$$

Now since f_2 is nondecreasing in x', independent of l' and f_1 is independent of z', we have

$$\tilde{f}_1^2(s,x,z,l) = \mathbb{E}^{'}[f_1(s,(X_s^{1,1})',(l_s^{1,1})',x,z,l)] \tag{VII.37}$$
$$\le \mathbb{E}^{'}[f_2(s,(X_s^{1,1})',(Z_s^{2,1})',x,z,l)] \tag{VII.38}$$
$$\le \mathbb{E}^{'}[f_2(s,(X_s^{2,1})',(Z_s^{2,1})',x,z,l)] = \tilde{f}_2^2(s,x,z,l) \tag{VII.39}$$

where the last inequality follows from (VII.36). Using again the comparison results for classic BSDEs with jumps again, we get

$$X_s^{1,2} \le X_s^{2,2}, \quad s \in [0,T].$$

By the same argument above, we can iteratively get that

$$X_s^{1,n} \le X_s^{2,n}, \quad s \in [0,T]. \quad n \ge 1.$$

We now show that for $i = 1,2$, $(X^{i,n}, Z^{i,n}, l^{i,n})_{n\ge 0}$ is Cauchy sequence in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2$. Put $\bar{X}_t^{i,n} = X_t^{i,n} - X_t^{i,n-1}$; $\bar{Z}_t^{i,n} = Z_t^{i,n} - Z_t^{i,n-1}$; $\bar{l}_t^{i,n} = l_t^{i,n} - l_t^{i,n-1}$, applying Ito's formula to $e^{\beta s}|X_s^{i,n} - X_s^{i,n-1}|^2$, $n \ge 1$, we have analogously to the Proposition A.4 [11] by properly choosing $\eta = \frac{1}{(T+2)8C^2}$,

$$\|\bar{X}^{i,n}\|_\beta^2 + \|\bar{Z}^{i,n}\|_\beta^2 + \|\bar{l}^{i,n}\|_{\nu,\beta}^2 \le \eta(T+2)4C^2(\|\bar{X}^{i,n-1}\|_\beta^2 + \|\bar{Z}^{i,n-1}\|_\beta^2 + \|\bar{l}^{i,n-1}\|_{\nu,\beta}^2) \tag{VII.40}$$
$$= \frac{1}{2}(\|\bar{X}^{i,n-1}\|_\beta^2 + \|\bar{Z}^{i,n-1}\|_\beta^2 + \|\bar{l}^{i,n-1}\|_{\nu,\beta}^2) \tag{VII.41}$$

If we denote the bound of $\|\bar{X}^{i,1}\|_\beta^2 + \|\bar{Z}^{i,1}\|_\beta^2 + \|\bar{l}^{i,1}\|_{\nu,\beta}^2$ by M, then by iteration we have,

$$\|\bar{X}^{i,k}\|_\beta^2 + \|\bar{Z}^{i,k}\|_\beta^2 + \|\bar{l}^{i,k}\|_{\nu,\beta}^2 \le \frac{M}{2^{k-1}}$$

which means that $(X^{i,n}, Z^{i,n}, l^{i,n})$ converges in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2$ to some (X^i, Z^i, l^i). Now taking limit in (VII.33), we see that (X^i, Z^i, l^i) is the unique solution of (VII.3). And $X_t^1 \le X_t^2, t \in [0,T] a.s.$ follows directly from the fact that $X_t^{1,n} \le X_t^{2,n}, t \in [0,T] a.s.$ □

Proofs of Theorem 1.2 Proof. Put $\bar{X}_t = X_t^1 - X_t^2$; $\bar{Z}_t = Z_t^1 - Z_t^2$; $\bar{l}_t(u) = l_t^1(u) - l_t^2(u)$; We then suppose that f_1 is independent of both z' and l', f_2 is nondecreasing in x'. We furthermore

Assumption 3.2 with strict inequality holds for f_1. Then

$$-dX_t = h_t dt - \bar{Z}_t dW_t - \int_{\mathbf{E}} \bar{l}_t(e)\tilde{N}(dt,de). \quad \bar{X}_T = \xi_1 - \xi_2.$$

where $h_t := \mathbb{E}'\left[f_1(t,\omega,X_t^{1'},X_t^1,Z_t^1,l_t^1(\cdot))\right] - \mathbb{E}'\left[f_2(t,\omega,X_t^{2'},Z_t^{2'},l_t^{2'}(\cdot),X_t^2,Z_t^2,l_t^2(\cdot))\right]$. Let $\phi(t) := \mathbb{E}'[f_1(t,X_t^{1'},X_t^2,Z_t^2,l_t^2)] - \mathbb{E}'[f_2(t,X_t^{2'},Z_t'^2,l_t'^2,X_t^2,Z_t^2,l_t^2)]$.

We have $h_t = \phi_t + \mathbb{E}'[f_1(t,X_t^{1'},X_t^1,Z_t^1,l_t^1) - f_1(s,X_t^{1'},X_t^2,Z_t^2,l_t^2)]$

We can write $f_1(t,X_t^{1'},X_t^1,Z_t^1,l_t^1) - f_1(t,X_t^{1'},X_t^2,Z_t^2,l_t^2) =$
$f_1(t,X_t^{1'},X_t^1,Z_t^1,l_t^1) - f_1(t,X_t^{1'},X_t^2,Z_t^1,l_t^1) + f_1(t,X_t^{1'},X_t^2,Z_t^1,l_t^1) - f_1(t,X_t^{1'},X_t^2,Z_t^2,l_t^1) + f_1(t,X_t^{1'},X_t^2,Z_t^2,l_t^1) - f_1(t,X_t^{1'},X_t^2,Z_t^2,l_t^2)$.

Then from Assumption 3.2 on f_1, there exists bounded processes δ and β on $\bar{\Omega}\times[0,T]$, such that

$$\mathbb{E}'f_1(t,X_t^{1'},X_t^1,Z_t^1,l_t^1) - \mathbb{E}'f_1(s,X_t^{1'},X_t^2,Z_t^2,l_t^2) \geq \mathbb{E}'[\delta_t]\bar{X}_t + \mathbb{E}'[\beta_t]\bar{Z}_t + \langle\theta_t,\bar{l}_t\rangle_\nu$$

with

$$\delta_t := \frac{f_1(t,X_t^{1'},X_t^1,Z_t^1,l_t^1) - f_1(t,X_t^{1'},X_t^2,Z_t^1,l_t^1)}{\bar{X}_t}$$

$$\beta_t := \frac{f_1(t,X_t^{1'},X_t^2,Z_t^1,l_t^1) - f_1(t,X_t^{1'},X_t^2,Z_t^2,l_t^1)}{\bar{Z}_t}$$

and θ_t is as in Assumption 3.2.
Thus we have $h_t \geq \phi_t + \mathbb{E}'[\delta_t]\bar{X}_t + \mathbb{E}'[\beta_t]\bar{Z}_t + \langle\theta_t,\bar{l}_t\rangle_\nu$. For each $t\in[0,T]$, let $(\Gamma_{t,s})_{s\in[t,T]}$ be the unique solution of the forward SDE

$$d\Gamma_{t,s} = \Gamma_{t,s^-}\left[\mathbb{E}'[\delta_s]ds + \mathbb{E}'[\beta_s]dW_s + \int_{\mathbf{E}}\theta_t(e)\tilde{N}(dt,de)\right]; \quad \Gamma_{t,t}=1.$$

By the comparison results with respect to a linear BSDE (see lemma 4.1 in [11]) we can derive that

$$\bar{X}_{t_0} \geq \mathbb{E}\left[\Gamma_{t_0,t}\bar{X}_t + \int_{t_0}^t \Gamma_{t_0,s}\phi(s)ds|\mathcal{F}_{t_0}\right], \quad t_0\leq t\leq T.$$

Due to the Theorem 1.1, we have $\bar{X}_t = X_t^1 - X_t^2 \geq 0$ and thus

$$f_1(t,X_t^{1'},X_t^2,Z_t^2,l_t^2) \geq f_2(s,X_t^{1'},Z_t'^2,l_t'^2,X_t^2,Z_t^2,l_t^2) \geq f_2(t,X_t^{2'},Z_t'^2,l_s'^2,X_t^2,Z_t^2,l_t^2)$$

by the assumption on f_1 and f_2. Then we can conclude the proof by pointing out the fact that

$$\phi(t) = \mathbb{E}'[f_1(t,X_t^{1'},X_t^2,Z_t^2,l_t^2)] - \mathbb{E}'[f_2(t,X_t^{2'},Z_t'^2,l_t'^2,X_t^2,Z_t^2,l_t^2)] \geq 0$$

and that if $\theta_t(u) > -1dP\otimes dt\otimes d\nu(u)$-a.s., then $\Gamma_{t,s}>0$ a.s. from Corollary 3.5 in [11]. □

Proofs of Theorem 2.1 Proof. Denote by $\mathbb{H}_\beta^2$ the space $\mathbb{H}^2\times\mathbb{H}^2\times\mathbb{H}_\nu^2$ equipped with the norm $\|Y,Z,k(\cdot)\|_\beta^2 := \|Y\|_\beta^2 + \|Z\|_\beta^2 + \|k\|_{\nu,\beta}^2$. We define a mapping Φ from $\mathbb{H}_\beta^2$ into itself as follows. Given $(U, V, l) \in \mathbb{H}^2_\beta$, let $(Y, Z, k) = \Phi(U, V, l)$ be the

the solution of the RBSDE associated with driver $f^1(s) := \mathbb{E}' f(s, U'_s, V'_s, l'_s, U_s, V_s, l_s)$.

Let A be the associated nondecreasing process. The mapping Φ is well defined by Theorem 1.1 .

Now we prove that the mapping Φ is a contraction from $\mathbb{H}^2_\beta$ into $\mathbb{H}^2_\beta$. Let $((\hat{U}, \hat{V}, \hat{l})$ be another element of $\mathbb{H}^2_\beta$ and let $(\hat{Y}, \hat{Z}, \hat{k}) := \Phi(\hat{U}, \hat{V}, \hat{l})$, that is, the solution of the RBSDE associated with driver process $\mathbb{E}' f(s, \hat{U}'_s, \hat{V}'_s, \hat{l}'_s, \hat{U}_s, \hat{V}_s, \hat{l}_s)$.

Set $\bar{U} = U - \hat{U}$, $\bar{V} = V - \hat{V}$, $\bar{l} = l - \hat{l}$, $\bar{Y} = Y - \hat{Y}$, $\bar{Z} = Z - \hat{Z}$, $\bar{k} = k - \hat{k}$.

Let $\Delta f. := \mathbb{E}' f(\cdot, U'_s, V'_s, l'_s, U_s, V_s, l_s) - \mathbb{E}' f(\cdot, \hat{U}'_s, \hat{V}'_s, \hat{l}'_s, \hat{U}_s, \hat{V}_s, \hat{l}_s)$. From Lipschitz continuity of f, we have $\mathbb{E}|\Delta f|^2 = \mathbb{E}|\mathbb{E}' f(\cdot, U'_s, V'_s, l'_s, U_s, V_s, l_s) - \mathbb{E}' f(\cdot, \hat{U}'_s, \hat{V}'_s, \hat{l}'_s, \hat{U}_s, \hat{V}_s, \hat{l}_s)|^2 \leq 6C^2 \mathbb{E}[|\overline{U}|^2 + |\overline{V}|^2 + \|\bar{l}\|^2_\nu]$. Recall that $\|\Delta f\|^2_\nu = [\int_0^T e^{\beta s} \mathbb{E}|\Delta f|^2 \, ds]$. Using estimates (A.58) and (A.59) in [12] with $\eta \leq \frac{1}{2C^2}$ and Lipschitz constant equal to 0 (since the driver f^1 does not depend on the solution), we get

$$\|\bar{Y}\|^2_\beta + \|\bar{Z}\|^2_\beta + \|\bar{k}\|^2_{\nu,\beta} \leq \eta(T+2)\|\Delta f\|^2_\beta \leq \eta(T+2) 6C^2 (\|\bar{U}\|^2_\beta + \|\bar{V}\|^2_\beta + \|\bar{l}\|^2_{\nu,\beta}),$$

Choosing $\eta = \frac{1}{(T+2)12C^2}$, we deduce $\|(\overline{Y}, \overline{Z}, \bar{k})\|^2_\beta \leq \frac{1}{2}\|(\overline{U}, \overline{V}, \bar{l})\|^2_\beta$. Hence, Φ is a contraction and thus admits a unique fixed point (Y, Z, k) in $\mathbb{H}^2_\beta$, which is the solution of meanfield RBSDE (VII.5).

For the second assertion of the continuity of A, we apply the same proof as in Theorem 1.2. $\square$

We provide below some a priori estimates which are used in the proof of Proposition 2.5. For $\beta > 0$ and $\phi \in \mathbb{H}^{2,T}$, we introduce the norm $\|\phi\|^2_{\beta,T} := E[\int_0^T e^{\beta s} \phi_s^2 ds]$ and for $l \in \mathbb{H}^{2,T}_\nu$, we set $\|l\|^2_{\nu,\beta,T} := E[\int_0^T e^{\beta s} \|l_s\|^2_\nu \, ds]$.

Apriori estimates for second-type mean-field BSDE

Proposition 4.1. *(Apriori estimates for mean-field BSDE with jumps) Let $T > 0$ and let $\xi \in \mathcal{S}^2$. Let f^1 be a Lipschitz driver with Lipschitz constant C and let f^2 be a driver. For $i = 1, 2$, let (Y^i, Z^i, k^i, A^i) be a solution of the Mean-field RBSDE associated to terminal time T, driver f^i and obstacle ξ. For s in $[0, T]$, denote $\bar{Y}_s := Y^1_s - Y^2_s$, $\bar{Z}_s := Z^1_s - Z^2_s$, $\bar{k}_s := k^1_s - k^2_s$, and $\bar{f}(s) := \mathbb{E}' f^1(s, Y^{2'}_s, Z^{2'}_s, k^{2'}_s, Y^2_s, Z^2_s, k^2_s) - \mathbb{E}' f^2(s, Y^{2'}_s, Z^{2'}_s, k^{2'}_s, Y^2_s, Z^2_s, k^2_s)$. Let $\eta, \beta > 0$ be such that $\beta \geq \frac{5}{\eta} + 4C$. If $\eta \leq \frac{1}{C^2}$, then, for each $t \in [0, T]$, we have*

$$e^{\beta t} \bar{Y}_t^2 \leq \eta \, E[\int_t^T e^{\beta s} \bar{f}(s)^2 ds \mid \mathcal{F}_t] \quad \text{a .s.} \tag{VII.42}$$

$$\|\bar{Y}\|^2_\beta \leq T\eta \|\bar{f}\|^2_\beta. \tag{VII.43}$$

Also, if $\eta < \frac{1}{C^2}$, we then have

$$\|\bar{Z}\|^2_\beta + \|\bar{k}\|^2_{\nu,\beta} \leq \frac{\eta}{1 - \eta C^2} \|\bar{f}\|^2_\beta. \tag{VII.44}$$

Proof. From Itô's formula applied to the semimartingale $e^{\beta s}\bar{Y}_s^2$ between t and T, it follows

$$\begin{aligned}
e^{\beta t}\bar{Y}_t^2 &+ \beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}\bar{Z}_s^2 ds + \int_t^T e^{\beta s}\|\bar{k}_s\|_\nu^2 ds + \sum_{t<s\le T} e^{\beta s}(\Delta A_s^1 - \Delta A_s^2)^2 \\
&= 2\int_t^T e^{\beta s}\bar{Y}_s[\mathbb{E}'f^1(s, Y_s^{1'}, Z_s^{1'}, k_s^{1'}, Y_s^1, Z_s^1, k_s^1) - \mathbb{E}'f^2(s, Y_s^{2'}, Z_s^{2'}, k_s^{2'}, Y_s^2, Z_s^2, k_s^2)]ds \\
&\quad - 2\int_t^T e^{\beta s}\bar{Y}_s\bar{Z}_s dW_s - \int_t^T e^{\beta s}\int_{\mathbb{R}^*}(2\bar{Y}_{s^-}\bar{k}_s(u) + \bar{k}_s(u)^2)d\tilde{N}(du, dt) \\
&\quad + 2\int_t^T e^{\beta s}\overline{Y}_{s^-}dA_s^1 - 2\int_t^T e^{\beta s}\overline{Y}_{s^-}dA_s^2 \qquad \text{(VII.45)}
\end{aligned}$$

Now, we have a.s.

$$\overline{Y}_s dA_s^{1,c} = (Y_s^1 - \xi_s)dA_s^{1,c} - (Y_s^2 - \xi_s)dA_s^{1,c} = -(Y_s^2 - \xi_s)dA_s^{1,c} \le 0$$

and by symmetry, $\overline{Y}_s dA_s^{2,c} \ge 0$ a.s. Also, we have a.s.

$$\overline{Y}_{s^-}\Delta A_s^{1,d} = (Y_{s^-}^1 - \xi_{s^-})\Delta A_s^{1,d} - (Y_{s^-}^2 - \xi_{s^-})\Delta A_s^{1,d} = -(Y_{s^-}^2 - \xi_{s^-})\Delta A_s^{1,d} \le 0$$

and $\overline{Y}_{s^-}\Delta A_s^{2,d} \ge 0$ a.s. Consequently, the two last terms of the r.h.s. of (VII.45) are non positive. Taking the conditional expectation given $\mathcal{F}_t$, we get

$$\begin{aligned}
e^{\beta t}\bar{Y}_t^2 &+ \mathbb{E}\left[\beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2)ds \mid \mathcal{F}_t\right] \\
&\le 2\mathbb{E}\left[\int_t^T e^{\beta s}\bar{Y}_s\mathbb{E}'[f^1(s, Y_s^{1'}, Z_s^{1'}, k_s^{1'}, Y_s^1, Z_s^1, k_s^1) - \mathbb{E}'f^2(s, Y_s^{2'}, Z_s^{2'}, k_s^{2'}, Y_s^2, Z_s^2, k_s^2)]ds \mid \mathcal{F}_t\right]. \qquad \text{(VII.46)}
\end{aligned}$$

Moreover,

$$\begin{aligned}
&|\mathbb{E}'f^1(s, Y_s^{1'}, Z_s^{1'}, k_s^{1'}, Y_s^1, Z_s^1, k_s^1) - \mathbb{E}'f^2(s, Y_s^{2'}, Z_s^{2'}, k_s^{2'}, Y_s^2, Z_s^2, k_s^2)| \\
&\le |\mathbb{E}'f^1(s, Y_s^{1'}, Z_s^{1'}, k_s^{1'}, Y_s^1, Z_s^1, k_s^1) - \mathbb{E}'f^1(s, Y_s^{2'}, Z_s^{2'}, k_s^{2'}, Y_s^2, Z_s^2, k_s^2)| + |\bar{f}_s| \\
&\le C(\mathbb{E}'|\bar{Y}_s^{'}| + \mathbb{E}'|\bar{Z}_s^{'}| + \mathbb{E}'\|\bar{k}_s^{'}\|_\nu) + C(|\bar{Y}_s| + |\bar{Z}_s| + \|\bar{k}_s\|_\nu) + |\bar{f}_s|.
\end{aligned}$$

Now, for all real numbers y, z', k', z, k, f and $\varepsilon > 0$
$2y(Cz' + Ck' + Cz + Ck + f) \le \frac{y^2}{\varepsilon^2} + \varepsilon^2(Cz' + Ck' + Cz + Ck + f)^2 \le \frac{y^2}{\varepsilon^2} + 5\varepsilon^2(C^2z'^2 + C^2k'^2 + C^2z^2 + C^2k^2 + f^2)$.

And that

$$\mathbb{E}'|\bar{Y}_s^{'}| = \mathbb{E}|\bar{Y}_s|, \mathbb{E}'|\bar{Z}_s^{'}| = \mathbb{E}|\bar{Z}_s|, \mathbb{E}'\|\bar{k}_s^{'}\|_\nu = \mathbb{E}\|\bar{k}_s\|_\nu,$$

so that $\mathbb{E}(|\bar{Y}_s||\mathbb{E}'|\bar{Y}_s^{'}|) = (\mathbb{E}|\bar{Y}_s|)^2 \le \mathbb{E}|\bar{Y}_s|^2$ and $(\mathbb{E}'|\bar{Z}_s^{'}|)^2 \le \mathbb{E}|\bar{Z}_s|^2$, $(\mathbb{E}'\|\bar{k}_s^{'}\|_\nu)^2 \le \mathbb{E}\|\bar{k}_s\|_\nu^2$. Hence, we get

$$e^{\beta t}\bar{Y}_t^2 + E\left[\beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2)ds \mid \mathcal{F}_t\right]$$
$$\leq E\left[(4C + \frac{1}{\varepsilon^2})\int_t^T e^{\beta s}\bar{Y}_s^2 ds + 5C^2\varepsilon^2\int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2)ds \mid \mathcal{F}_t\right]$$
$$+ 5\varepsilon^2 E\left[\int_t^T e^{\beta s}\bar{f}_s^2 ds \mid \mathcal{F}_t\right]. \tag{VII.47}$$

Let us make the change of variable $\eta = 5\epsilon^2$. Then, for each $\beta, \eta > 0$ chosen as in the proposition, these inequalities lead to (VII.42). We obtain the first inequality of (VII.43) by integrating (VII.42). Then (VII.44) follows from inequality (VII.47). □

Propositions 2.2 to Propositions 3.4 The proofs of Propositions 2.2 to Propositions 3.4 use the same idea as their counterpart (Theorem/Proposition 2.1 to 1.5) with the replacement of comparison theory Th.1.2, Th.1.3 and prior estimates (4.3) by the cutting edged counterpart Th.1.1, Th.1.2 and prior estimates (4.1) for the second type.

Bibliography

[1] R. Carmona, J.-P. Fouque, and L.-H. Sun, "Mean field games and systemic risk," *arXiv preprint arXiv:1308.2172, To appear in Communications in Mathematical Sciences*, 2013.

[2] Dellacherie, C. and Meyer, P.-A. (1975): *Probabilité et Potentiel, Chap. I-IV.* Nouvelle édition. Hermann. **MR**0488194

[3] El Karoui, Nicole and Pardoux, E and Quenez, Marie Claire, "Reflected backward SDEs and American options", *Numerical methods in finance*, vol. 13, pp. 215–231, 1997.

[4] J. Garnier, G. Papanicolaou, and T.-W. Yang, "Large deviations for a mean field model of systemic risk," *SIAM Journal on Financial Mathematics*, vol. 4, no. 1, pp. 151–184, 2013.

[5] J. Garnier, G. Papanicolaou, and T.-W. Yang, "Diversification in financial networks may increase systemic risk," *Handbook on Systemic Risk*, p. 432, 2013.

[6] J.-P. Fouque and T. Ichiba, "Stability in a model of interbank lending," *SIAM Journal on Financial Mathematics*, vol. 4, no. 1, pp. 784–803, 2013.

[7] J-B.Hiriart-Urruty and C.Lemarechal : Convex Analysis and Minimization Algorithms II *Grundlehren der mathematischen Wissenchaften 306*

[8] J.Li and H.Min : Controlled mean-field backward stochastic differential equations with jumps involving the value function, *Journal of Systems Science and Complexity* (2016), 1–31.

[9] Buckdahn, R., Djehiche, B., Li, J., Peng, S.: Mean-field backward stochastic differential equations. A limit approach. *The Annals of Probability* 37(4)(2009), 1524–1565

[10] Buckdahn, R., Li, J. and Peng, S.: Mean-field backward stochastic differential equations and related partial differential equations. *Stochastic Processes and Their Applications* 119(2009), 3133–3154.

[11] Quenez M-C. and Sulem A. : BSDEs with jumps, optimization and applications to dynamic risk measures. *Stochastic Processes and Their Applications* 123 (2013), 0–29.

[12] Quenez M-C. and Sulem A. : Reflected BSDEs and robust optimal stopping for dynamic risk measures with jumps. *Stochastic Processes and Their Applications* 124 (2014), 3031–3054.

[13] M.Royer : Backward stochastic differential equations with jumps and related non-linear expectations. *Stochastic Processes and Their Applications* 116 (2006), 1358–1376.

[14] N.Agram, Y-Z.Hu, B.Oksendall : Mean-field backward stochastic differential equations and applications. *https://arxiv.org/pdf/1801.03349.pdf*

[15] Hamadene, S., Ouknine, Y. : Reflected backward stochastic differential equation with jumps and random obstacle. *Electronic Journal of Probability* 8, 1?20, 2003.

Part 4

Optimal inventory management and order book modeling

In Section VIII, we present the order book dynamics and illustrate the MM control problem. Still in this section, we present the equations satisfied by the MM optimal strategy and propose a numerical solution for this problem. Next, in Section IX, we formulate the HFT control problem using a pair trading strategy. We provide the modeling framework and the equation satisfied the HFT impulse-control strategy as well as the numerical solution for this problem. Furthermore, we present an IB optimal liquidation strategy using the VWAP price as a benchmark. We solve the broker execution problem in a continuous-time framework. But we also present a discrete time setting for a simpler implementation. Finally, in the last section, we provide a methodology to simulate a realistic market using the three agent's optimal trading strategies.

CHAPTER VIII

Market maker strategy

In this section, we describe the order book dynamics and the optimal control problem of the Market Maker, the key tools to characterize the solution and how to numerically approximate this solution.

1 The order book dynamics

All over this paper, for $d \geq 0$, $D([0,T];\mathbb{R}^d)$ is the Skorohod space of càdlàg functions from $[0,T]$ into $\mathbb{R}^d$ and $\mathbb{P}$ a probability measure on this space. We denote respectively $(P^{\mathfrak{b}})_{t\geq 0}$ and $(P^{\mathfrak{a}})_{t\geq 0}$ the best bid offer and the best ask offer processes on the market, valued in $\delta\mathbb{Z}$ where $\delta > 0$ is the tick size. We denote $(Q^{\mathfrak{b}})_{t\geq 0}$ and $(Q^{\mathfrak{a}})_{t\geq 0}$ the size of the corresponding queues valued in $\mathbb{N}^*$. For ease of notation, we introduce $P := (P^{\mathfrak{b}}, P^{\mathfrak{a}})$ and $Q := (Q^{\mathfrak{b}}, Q^{\mathfrak{a}})$ and $S := P^{\mathfrak{a}} - P^{\mathfrak{b}}$ the tick size process. We shall assume that, $\mathbb{P}$-a.s., for all $t \geq 0$, $S_t \in \{\delta, 2\delta\}$.

The prices P and the queues Q will evolve according to the arrival of several types of order coming from a random Poisson measure. We first define these orders:

- Agressive orders of size $\alpha^{\mathfrak{b}} \in \mathbb{N} \cap [0, Q^{\mathfrak{b}}]$ at the bid or of size $\alpha^{\mathfrak{a}} \in \mathbb{N} \cap [0, Q^{\mathfrak{a}}]$ at the ask: the size of corresponding queue, $Q^{\mathfrak{b}}$ or $Q^{\mathfrak{a}}$, decreases by the size of the aggressive order, $\alpha^{\mathfrak{b}}$ or $\alpha^{\mathfrak{a}}$.

- Limit orders of size $L^{\mathfrak{b}} \in \mathbb{N}$ at the bid or of size $L^{\mathfrak{a}} \in \mathbb{N}$ at the ask: the size of corresponding queue, $Q^{\mathfrak{b}}$ or $Q^{\mathfrak{a}}$, increases by the size of the aggressive order, $L^{\mathfrak{b}}$ or $L^{\mathfrak{a}}$.

- On $\{S = 2\delta\}$: Limit orders of size $L^{\mathfrak{b},\frac{1}{2}} \in \mathbb{N}$ at the bid or of size $L^{\mathfrak{a},\frac{1}{2}} \in \mathbb{N}$ the ask: the order is place inside the spread, at the price $P^{\mathfrak{b}} + \delta = P^{\mathfrak{a}} - \delta$, this generates a new queue at the bid or at the ask, of size $L^{\mathfrak{b},\frac{1}{2}}$ or $L^{\mathfrak{a},\frac{1}{2}}$, and a price move.

An aggressive order can set a Queue to 0. In this case, this queue might be regenerated ; or a new queue at the price of a tick less (at the bid) or of a tick more (at the ask) can be generated. If the price, for example at the bid, decreases and if the tick size was 2, then a new queue is generated at the ask with the price also reduced by one tick in order to keep the max two-tick size.

We are now in position to describe the random Poisson measure which will drive our processes. Let $N(dt, du)$ be a random Poisson measure on $E := \mathbb{N}^9$ and let be $\nu(P, Q; dt, de)$ be the finite associated compensator and $(\mathcal{F}_t)_{t\geq 0}$ the generated filtration. We assume that only one type of order can arrive at the same time. When a jump of size $e =$

$(\alpha^{\mathfrak{b}}, \alpha^{\mathfrak{a}}, L^{\mathfrak{b}}, L^{\mathfrak{a}}, L^{\mathfrak{b},\frac{1}{2}}, L^{\mathfrak{a},\frac{1}{2}}, Z, \varepsilon^{\mathfrak{b}}, \varepsilon^{\mathfrak{a}})$ occurs, then:

$$\begin{aligned}
P^{\mathfrak{b}}_t &= P^{\mathfrak{b}}_{t-} + \delta \left[\mathbf{1}_{\{Z=1\}} \left(-\mathbf{1}_{\{Q^{\mathfrak{b}}_{t-}=\alpha^{\mathfrak{b}}\}} + \mathbf{1}_{\{P^{\mathfrak{a}}_t - P^{\mathfrak{b}}_t = 2\delta\}} \mathbf{1}_{\{Q^{\mathfrak{a}}_{t-}=\alpha^{\mathfrak{a}}\}}\right) + \mathbf{1}_{\{L^{\mathfrak{b},\frac{1}{2}}>0\}}\right] \\
P^{\mathfrak{a}}_t &= P^{\mathfrak{a}}_{t-} + \delta \left[\mathbf{1}_{\{Z=1\}} \left(\mathbf{1}_{\{Q^{\mathfrak{a}}_{t-}=\alpha^{\mathfrak{a}}\}} - \mathbf{1}_{\{P^{\mathfrak{a}}_t - P^{\mathfrak{b}}_t = 2\delta\}} \mathbf{1}_{\{Q^{\mathfrak{b}}_{t-}=\alpha^{\mathfrak{b}}\}}\right) - \mathbf{1}_{\{L^{\mathfrak{a},\frac{1}{2}}>0\}}\right] \\
Q^{\mathfrak{b}}_t &= Q^{\mathfrak{b}}_{t-} + (\varepsilon^{\mathfrak{b}} - Q^{\mathfrak{b}}_{t-}) \max\left\{\mathbf{1}_{\{Q^{\mathfrak{b}}_{t-}=\alpha^{\mathfrak{b}}\}}; \mathbf{1}_{\{Q^{\mathfrak{a}}_{t-}=\alpha^{\mathfrak{a}}\}}\right\} + L^{\mathfrak{b}} + (L^{\mathfrak{b},\frac{1}{2}} - Q^{\mathfrak{b}}_{t-})\mathbf{1}_{\{L^{\mathfrak{b},\frac{1}{2}}\}>0} \\
Q^{\mathfrak{a}}_t &= Q^{\mathfrak{a}}_{t-} + (\varepsilon^{\mathfrak{a}} - Q^{\mathfrak{a}}_{t-}) \max\left\{\mathbf{1}_{\{Q^{\mathfrak{a}}_{t-}=\alpha^{\mathfrak{a}}\}}; \mathbf{1}_{\{Q^{\mathfrak{b}}_{t-}=\alpha^{\mathfrak{b}}\}}\right\} + L^{\mathfrak{a}} + (L^{\mathfrak{a},\frac{1}{2}} - Q^{\mathfrak{b}}_{t-})\mathbf{1}_{\{L^{\mathfrak{a},\frac{1}{2}}\}>0}
\end{aligned} \tag{VIII.1}$$

Here, $Z = 1$ indicates that, if the queue size is 0 after an aggressive order, a price move occurs. The random variables $\varepsilon^{\mathfrak{b}}$ and $\varepsilon^{\mathfrak{a}}$ represent the size of the regenerated queue. The relation above are consistent since we assume than there is at top one type of order when a jump occurs. Here after we will assume $Q^{\mathfrak{a}}$ and $Q^{\mathfrak{b}}$ are bounded by some integer Q^*.

2 The Market Maker dynamics and Control set

The Market Maker will place orders in the orderbook in order to have, in average, a gain. Then, we have to record:

- His cash, $(G_t)_{t\geq 0}$;
- His inventory, $(I_t)_{t\geq 0}$;
- The size of placed limit order at bid queue, $(N^{\mathfrak{b}})_{t\geq 0}$, and at the ask queue, $N^{\mathfrak{a}}$, let $N := (N^{\mathfrak{b}}, N^{\mathfrak{a}})$;
- The position of the placed order at bid queue, $B^{\mathfrak{b}}$, and at the ask queue, $B^{\mathfrak{a}}$, let $B := (B^{\mathfrak{b}}, B^{\mathfrak{a}})$.

Let $\mathbf{A} \subset \mathbb{N}^6$ be a bounded set. A control is a sequence of random variables $\zeta = (\tau_i, \mathbf{a}_i)_{i\geq 1}$ where, for all $i \geq 1$,

- τ_i is a $\mathcal{F}$-stopping time ;
- $\mathbf{a}_i := (\alpha_i^{\mathfrak{b},M}, \alpha_i^{\mathfrak{a},M}, L_i^{\mathfrak{b},M}, L_i^{\mathfrak{a},M}, L_i^{\mathfrak{b},\frac{1}{2},M}, L_i^{\mathfrak{a},\frac{1}{2},M})$ is a $\mathcal{F}_{\tau_i}$-measurable random variable valued in $\mathbf{A}$;
- $\mathbf{1}_{\{S=1\}}(L_i^{\mathfrak{b},\frac{1}{2},M} + L_i^{\mathfrak{a},\frac{1}{2},M}) + (\alpha_i^{\mathfrak{b},M} + \alpha_i^{\mathfrak{a},M})(L_i^{\mathfrak{b},M} + L_i^{\mathfrak{a},M} + L_i^{\mathfrak{b},\frac{1}{2},M} + L_i^{\mathfrak{a},\frac{1}{2},M}) + L_i^{\mathfrak{b},M} L_i^{\mathfrak{b},\frac{1}{2},M} + L_i^{\mathfrak{a},M} L_i^{\mathfrak{a},\frac{1}{2},M} + \alpha_i^{\mathfrak{b},M}\alpha_i^{\mathfrak{a},M} = 0$

We denote by Φ° the set of controls. We will constraint the Market Maker to have a minimum and maximum inventory size, denoted by $(I_*, I^*) \in \mathbb{Z}^2$. A control will be admissible if, moreover, for all $i \geq 1$,

- $\alpha_i^{\mathfrak{b},M} \leq \min\left\{I_{\tau_i-} - I_* - B^{\mathfrak{a}}_{\tau_i-}; Q^{\mathfrak{b}}_{\tau_i-}\right\}$;
- $\alpha_i^{\mathfrak{a},M} \leq \min\left\{I^* - I_{\tau_i-} - B^{\mathfrak{b}}_{\tau_i-}; Q^{\mathfrak{a}}_{\tau_i-}\right\}$;
- $L_i^{\mathfrak{b},M} \leq \min\left\{I^* - I_{\tau_i-}; Q^* - Q^{\mathfrak{b}}_{\tau_i-}\right\}$;

- $L_i^{\mathfrak{a},M} \leq \min\left\{I_{\tau_i-} - I_*; Q^* - Q^{\mathfrak{a}}_{\tau_i-}\right\}$;
- $L_i^{\mathfrak{b},\frac{1}{2},M} \leq \min\left\{I^* - I_{\tau_i-}; Q^*\right\}$;
- $L_i^{\mathfrak{a},\frac{1}{2},M} \leq \min\left\{I_{\tau_i-} - I_*; Q^*\right\}$;

and we denote by Φ the corresponding set.

For a control $\phi \in \Phi$, and for $x = (g, i, n, b, p, q) \in \delta\mathbb{Z} \times \mathbb{N} \times (\mathbb{N} \cup \{\infty\})^2 \times \mathbb{N}^2 \times (\delta\mathbb{Z})^2 \times \mathbb{N}^2$, we can describe the dynamics of the processes associated to the Market Maker:

$$G^{t,x} := g + \sum_{i\geq 1} \mathbf{1}_{\{t\leq\tau_i<\cdot\}}(\alpha_i^b P_{\tau_i}^{\mathfrak{b},t,x} - \alpha_i^a P_{\tau_i}^{\mathfrak{a},t,x}) + \int_t^s \left(P_{u-}^{t,x} \cdot \beta(B_{u-}^{t,x}, N_{u-}^{t,x}, e)^T\right) N(du, de)$$

$$I^{t,x} := i + \sum_{i\geq 1} \mathbf{1}_{\{t\leq\tau_i<\cdot\}}(\alpha_i^a - \alpha_i^b) - \int_t^s \left(1 \cdot \beta(B_{u-}^{t,x}, N_{u-}^{t,x}, e)^T\right) N(du, de)$$

$$N^{t,x} := n + \sum_{i\geq 1} \mathbf{1}_{\{t\leq\tau_i<\cdot, L_i + L_i^{\frac{1}{2}}>0\}}(L_i + L_i^{\frac{1}{2}} - N_{\tau_i-}^{t,x}) - \int_t^s |\beta(B_{u-}^{t,x}, N_{u-}^{t,b}, e)| N(du, de)$$

$$B^{t,x} := b + \sum_{i\geq 1} \mathbf{1}_{\{t\leq\tau_i<\cdot, L_i + L_i^{\frac{1}{2}}>0\}}(Q_{\tau_i} - B_{\tau_i-}^{t,x}) - \int_t^s \max(\alpha, B_{u-}^{t,b}) N(du, de)$$

With

$$\beta(b, n, e) : \mathbb{N} \times \mathbb{N} \times E \to \mathbb{N}^2$$
$$(b, n, e) \mapsto (-\min\{(\alpha^{\mathfrak{b}} - b)^+, n\}, \min\{(\alpha^{\mathfrak{a}} - b)^+, n\})$$

Moreover, the processes P and Q evolve at the τ_i using the same rules as in (VIII.1). In the sequel, we shall simply write

$$X := (G, I, N, B, P, Q),$$

whose values belongs to $\mathbf{X} := \delta\mathbb{Z} \times \mathbb{N} \times (\mathbb{N} \cup \{\infty\})^2 \times \mathbb{N}^2 \times (\delta\mathbb{Z})^2 \times \mathbb{N}^2$. Now we give the transition state of the controlled Markov processes X^ζ.

3 Transition states of the reduced state process X^ζ

Exogenous market participants influence: Let Q the infinitesimal generator matrix of the continuous Markov jump process Y_t^ζ. For each $x = (g, i, n^{\mathfrak{b}}, n^{\mathfrak{a}}, b^{\mathfrak{b}}, b^{\mathfrak{a}}, p^{\mathfrak{a}}, p^{\mathfrak{b}}, q^{\mathfrak{b}}, q^{\mathfrak{a}}) \in \delta\mathbb{Z} \times \mathbb{N} \times (\mathbb{N} \cup \{\infty\})^2 \times \mathbb{N}^2 \times (\delta\mathbb{Z})^2 \times \mathbb{N}^2$, we have

- In the case of a bid limit order of size k,
  $x' = (g, i, n^{\mathfrak{b}}, n^{\mathfrak{a}}, b^{\mathfrak{b}}, b^{\mathfrak{a}}, p^{\mathfrak{a}}, p^{\mathfrak{b}}, q^{\mathfrak{b}} + k, q^{\mathfrak{a}})$ and

$$Q(x, x') = \lambda_L^{x,\mathfrak{b}}(k).$$

- In the case an ask limit order of size k,
  $x' = (g, i, n^{\mathfrak{b}}, n^{\mathfrak{a}}, b^{\mathfrak{b}}, b^{\mathfrak{a}}, p^{\mathfrak{a}}, p^{\mathfrak{b}}, q^{\mathfrak{b}}, q^{\mathfrak{a}} + k)$ and

$$Q(x, x') = \lambda_L^{x,\mathfrak{a}}(k).$$

- In the case of a bid limit order insertion within the spread of size k, $x' = (g, i, n^{\mathfrak{b}}, \infty, b^{\mathfrak{b}}, 0, p^{\mathfrak{b}}, p^{\mathfrak{a}} - \delta, q^{\mathfrak{b}}, k)$ and

$$Q(x, x') = \lambda^{x,\mathfrak{b}}_{L,\frac{1}{2}}(k)\mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} > \delta\}}.$$

- In the case of an ask limit order insertion within the spread of size k, $x' = (g, i, n^{\mathfrak{b}}, \infty, b^{\mathfrak{b}}, 0, p^{\mathfrak{b}}, p^{\mathfrak{a}} - \delta, q^{\mathfrak{b}}, k)$ and

$$Q(x, x') = \lambda^{x,\mathfrak{a}}_{L,\frac{1}{2}}(k)\mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} > \delta\}}$$

Recall that $R^{\mathfrak{b}}(x; .)$ (resp. $R^{\mathfrak{a}}(x; .)$) denotes the regenerated size on the new limits when the previous best bid (resp. ask) is completely depleted. For all $j \in \{\mathfrak{a}, \mathfrak{b}\}$, we note $R^j(x; .)^1$ (resp. $R^j(x; .)^2$) the new bid (resp. ask) limit.

- In the case of a bid market order of size k,
$x' = (g', i', n'^{\mathfrak{b}}, n'^{\mathfrak{a}}, b'^{\mathfrak{b}}, b'^{\mathfrak{a}}, p'^{\mathfrak{b}}, p'^{\mathfrak{a}}, q'^{\mathfrak{b}}, q'^{\mathfrak{a}})$ where

$$\begin{cases} g' = g + p^{\mathfrak{b}} \min\{(k - n^{\mathfrak{b}})^+, b^{\mathfrak{b}}\} - p^{\mathfrak{a}} b^{\mathfrak{a}} \\ i' = i + \min\{(k - n^{\mathfrak{b}} + 1)^+, b^{\mathfrak{b}}\} \\ n'^{\mathfrak{b}} = \max\{1, [n^{\mathfrak{b}} - k]^+\}\mathbf{1}_{\{n^{\mathfrak{b}} + b^{\mathfrak{b}} - 1 - k > 0\}} + \infty \mathbf{1}_{\{n^{\mathfrak{b}} + b^{\mathfrak{b}} - 1 - k \le 0\}} \\ n'^{\mathfrak{a}} = n^{\mathfrak{a}} \mathbf{1}_{\mathcal{A}} + \infty \mathbf{1}_{\mathcal{A}^c}, \qquad (\mathcal{A} = \{q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{a}} > 0\} \cup \{p^{\mathfrak{a}} - p^{\mathfrak{b}} = \delta\}) \\ b'^{\mathfrak{b}} = \max\{0, b^{\mathfrak{b}} - [k - n^{\mathfrak{b}} + 1]^+\} \\ b'^{\mathfrak{a}} = b^{\mathfrak{a}} \mathbf{1}_{\mathcal{A}} \\ p'^{\mathfrak{b}} = p^{\mathfrak{b}} \mathbf{1}_{\{q^{\mathfrak{a}} - q^{\mathfrak{b}} = \delta\}} + (p^{\mathfrak{b}} - \delta)\mathbf{1}_{\{q^{\mathfrak{a}} - q^{\mathfrak{b}} = 2\delta\}} \\ p'^{\mathfrak{a}} = p^{\mathfrak{a}} \mathbf{1}_{\{q^{\mathfrak{a}} - k > 0\}} + (p^{\mathfrak{a}} - \delta)\mathbf{1}_{\{q^{\mathfrak{a}} - k \le 0\}} \\ q'^{\mathfrak{b}} = (q^{\mathfrak{b}} - k)\mathbf{1}_{\{q^{\mathfrak{b}} - k > 0\}} + R^{\mathfrak{b}}(x; .)^1 \mathbf{1}_{\{q^{\mathfrak{b}} - k \le 0\}} \\ q'^{\mathfrak{a}} = q^{\mathfrak{a}} \mathbf{1}_{\{q^{\mathfrak{b}} - k > 0\}} + R^{\mathfrak{b}}(x; .)^2 \mathbf{1}_{\{q^{\mathfrak{b}} - k \le 0\}}, \end{cases}$$

and

$$Q(x, x') = \lambda^{x,\mathfrak{b}}_{A}(k).$$

- In the case of ask market order of size k,

$x' = (g', i', n'^{\mathfrak{b}}, n'^{\mathfrak{a}}, b'^{\mathfrak{b}}, b'^{\mathfrak{a}}, p'^{\mathfrak{b}}, p'^{\mathfrak{a}}, q'^{\mathfrak{b}}, q'^{\mathfrak{a}})$ where

$$\begin{cases} g' = g + p^{\mathfrak{b}} b^{\mathfrak{b}} - p^{\mathfrak{a}} \min\{(k - n^{\mathfrak{a}})^+, b^{\mathfrak{a}}\} \\ i' = i - \min\{(k - n^{\mathfrak{a}} + 1)^+, b^{\mathfrak{a}}\} \\ n'^{\mathfrak{b}} = n^{\mathfrak{b}} \mathbf{1}_{\mathcal{A}} + \infty \mathbf{1}_{\mathcal{A}^c}, \qquad (\mathcal{A} = \{q^{\mathfrak{b}} - \Delta\alpha^{\mathfrak{b}} > 0\} \cup \{p^{\mathfrak{a}} - p^{\mathfrak{b}} = \delta\}) \\ n'^{\mathfrak{a}} = \max\{1, [n^{\mathfrak{a}} - k]^+\} \mathbf{1}_{\{n^{\mathfrak{a}} + b^{\mathfrak{a}} - 1 - k > 0\}} + \infty \mathbf{1}_{\{n^{\mathfrak{a}} + b^{\mathfrak{a}} - 1 - k \le 0\}} \\ b'^{\mathfrak{b}} = b^{\mathfrak{b}} \mathbf{1}_{\mathcal{A}} \\ b'^{\mathfrak{a}} = \max\{0, b^{\mathfrak{a}} - [k - n^{\mathfrak{a}} + 1]^+\} \\ p'^{\mathfrak{b}} = p^{\mathfrak{b}} \mathbf{1}_{\{q^{\mathfrak{a}} - q^{\mathfrak{b}} = \delta\}} + (p^{\mathfrak{b}} - \delta) \mathbf{1}_{\{q^{\mathfrak{a}} - q^{\mathfrak{b}} = 2\delta\}} \\ p'^{\mathfrak{a}} = p^{\mathfrak{a}} \mathbf{1}_{\{q^{\mathfrak{a}} - k > 0\}} + (p^{\mathfrak{a}} - \delta) \mathbf{1}_{\{q^{\mathfrak{a}} - k \le 0\}} \\ q'^{\mathfrak{b}} = q^{\mathfrak{b}} \mathbf{1}_{\{q^{\mathfrak{a}} - k > 0\}} + R^{\mathfrak{a}}(x; .)^1 \mathbf{1}_{\{q^{\mathfrak{a}} - k \le 0\}} \\ q'^{\mathfrak{a}} = (q^{\mathfrak{a}} - k) \mathbf{1}_{\{q^{\mathfrak{a}} - k > 0\}} + R^{\mathfrak{a}}(x; .)^2 \mathbf{1}_{\{q^{\mathfrak{a}} - k \le 0\}}, \end{cases}$$

and

$$Q(x, x') = \lambda_A^{y, \mathfrak{a}}(k)$$

Controller influence: Now we define the map $\phi^\zeta(x)$ for all $\zeta \in \mathrm{Z}(x)$, where

$$\mathrm{Z}(x) = \{z \in \mathrm{Z}^\circ : z^1 \le q^{\mathfrak{b}}, z^2 \le q^{\mathfrak{a}}, z^3 \mathbf{1}_{b^{\mathfrak{b}} > 0} = 0, z^4 \mathbf{1}_{b^{\mathfrak{a}} > 0} = 0, (z^5 + z^6) \mathbf{1}_{p^{\mathfrak{a}} - p^{\mathfrak{b}} = \delta} = 0\},$$

and

$$\mathrm{Z}^\circ := \{e \in \mathbb{N}^6 : \sum_{i=3}^{6} (e^1 + e^2) e^i + e^1 e^2 + \sum_{i=3}^{4} e^i e^{i+2} = 0\}.$$

Let $y' := \phi^\zeta(x) = (i', n'^{\mathfrak{b}}, n'^{\mathfrak{a}}, b'^{\mathfrak{b}}, b'^{\mathfrak{a}}, p'^{\mathfrak{b}}, p'^{\mathfrak{a}}, q'^{\mathfrak{b}}, q'^{\mathfrak{a}})$

- In the case where the market maker places a bid market order of size $\Delta\alpha^{\mathfrak{b}}$. We have $\Delta\zeta \;= (\Delta\alpha^{\mathfrak{b}}, 0, 0, 0, 0, 0)$ and

$$\begin{cases} g' = g - p^{\mathfrak{b}} \left(\min\{\Delta\alpha^{\mathfrak{b}}, n^{\mathfrak{b}} - 1\} + [\Delta\alpha^{\mathfrak{b}} - (n^{\mathfrak{b}} + b^{\mathfrak{b}} - 1)]^+ \right) - p^{\mathfrak{a}} b^{\mathfrak{a}} \\ i' = i - \min\{\Delta\alpha^{\mathfrak{b}}, n^{\mathfrak{b}} - 1\} - [\Delta\alpha^{\mathfrak{b}} - (n^{\mathfrak{b}} + b^{\mathfrak{b}} - 1)]^+ \\ n'^{\mathfrak{b}} = \max\{1, [n^{\mathfrak{b}} - \Delta\alpha^{\mathfrak{b}}]^+\} \mathbf{1}_{\{n^{\mathfrak{b}} + b^{\mathfrak{b}} - 1 - \Delta\alpha^{\mathfrak{b}} > 0\}} + \infty \mathbf{1}_{\{n^{\mathfrak{b}} + b^{\mathfrak{b}} - 1 - \Delta\alpha^{\mathfrak{b}} \le 0\}} \\ n'^{\mathfrak{a}} = n^{\mathfrak{a}} \mathbf{1}_{\mathcal{A}} + \infty \mathbf{1}_{\mathcal{A}^c}, \qquad (\mathcal{A} = \{q^{\mathfrak{b}} - \Delta\alpha^{\mathfrak{b}} > 0\} \cup \{p^{\mathfrak{a}} - p^{\mathfrak{b}} = \delta\}) \\ b'^{\mathfrak{b}} = \max\{0, b^{\mathfrak{b}} - [\Delta\alpha^{\mathfrak{b}} - n^{\mathfrak{b}} + 1]^+\} \\ b'^{\mathfrak{a}} = b^{\mathfrak{a}} \mathbf{1}_{\mathcal{A}} \\ p'^{\mathfrak{b}} = p^{\mathfrak{b}} \mathbf{1}_{\{q^{\mathfrak{a}} - q^{\mathfrak{b}} = \delta\}} + (p^{\mathfrak{b}} - \delta) \mathbf{1}_{\{q^{\mathfrak{a}} - q^{\mathfrak{b}} = 2\delta\}} \\ p'^{\mathfrak{a}} = p^{\mathfrak{a}} \mathbf{1}_{\{q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{b}} > 0\}} + (p^{\mathfrak{a}} - \delta) \mathbf{1}_{\{q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{b}} \le 0\}} \\ q'^{\mathfrak{b}} = (q^{\mathfrak{b}} - \Delta\alpha^{\mathfrak{b}}) \mathbf{1}_{\{q^{\mathfrak{b}} - \Delta\alpha^{\mathfrak{b}} > 0\}} + R^{\mathfrak{b}}(x; .)^1 \mathbf{1}_{\{q^{\mathfrak{b}} - \Delta\alpha^{\mathfrak{b}} \le 0\}} \\ q'^{\mathfrak{a}} = q^{\mathfrak{a}} \mathbf{1}_{\{q^{\mathfrak{b}} - \Delta\alpha^{\mathfrak{b}} > 0\}} + R^{\mathfrak{b}}(x; .)^2 \mathbf{1}_{\{q^{\mathfrak{b}} - \Delta\alpha^{\mathfrak{b}} \le 0\}}. \end{cases}$$

- In the case where the market maker places an ask market order of size $\Delta\alpha^{\mathfrak{a}}$. We have

$\Delta\zeta \ = (0, \Delta\alpha^{\mathfrak{a}}, 0, 0, 0, 0)$ and

$$
\begin{cases}
g' = g + p^{\mathfrak{b}} b^{\mathfrak{b}} - p^{\mathfrak{a}} \left(\min\{\Delta\alpha^{\mathfrak{b}}, n^{\mathfrak{a}} - 1\} + [\Delta\alpha^{\mathfrak{a}} - (n^{\mathfrak{a}} + b^{\mathfrak{a}} - 1)]^+ \right) \\
i' = i + \min\{\Delta\alpha^{\mathfrak{a}}, n^{\mathfrak{a}} - 1\} + [\Delta\alpha^{\mathfrak{a}} - (n^{\mathfrak{a}} + b^{\mathfrak{a}} - 1)]^+ \\
n'^{\mathfrak{b}} = n^{\mathfrak{b}} \mathbf{1}_{\mathcal{A}} + \infty \mathbf{1}_{\mathcal{A}^c}, \qquad (\mathcal{A} = \{q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{a}} > 0\} \cup \{p^{\mathfrak{a}} - p^{\mathfrak{b}} = \delta\}) \\
n'^{\mathfrak{a}} = \max\{1, [n^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{a}}]^+\} \mathbf{1}_{\{n^{\mathfrak{a}} + b^{\mathfrak{a}} - 1 - \Delta\alpha^{\mathfrak{a}} > 0\}} + \infty \mathbf{1}_{\{n^{\mathfrak{a}} + b^{\mathfrak{a}} - 1 - \Delta\alpha^{\mathfrak{a}} \leq 0\}} \\
b'^{\mathfrak{b}} = b^{\mathfrak{b}} \mathbf{1}_{\mathcal{A}} \\
b'^{\mathfrak{a}} = \max\{0, b^{\mathfrak{a}} - [\Delta\alpha^{\mathfrak{a}} - n^{\mathfrak{a}} + 1]^+\} \\
p'^{\mathfrak{b}} = p^{\mathfrak{b}} \mathbf{1}_{\{q^{\mathfrak{a}} - q^{\mathfrak{b}} = \delta\}} + (p^{\mathfrak{b}} - \delta) \mathbf{1}_{\{q^{\mathfrak{a}} - q^{\mathfrak{b}} = 2\delta\}} \\
p'^{\mathfrak{a}} = p^{\mathfrak{a}} \mathbf{1}_{\{q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{a}} > 0\}} + (p^{\mathfrak{a}} - \delta) \mathbf{1}_{\{q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{a}} \leq 0\}} \\
q'^{\mathfrak{b}} = q^{\mathfrak{b}} \mathbf{1}_{\{q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{a}} > 0\}} + R^{\mathfrak{a}}(x; .)^1 \mathbf{1}_{\{q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{a}} \leq 0\}} \\
q'^{\mathfrak{a}} = (q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{a}}) \mathbf{1}_{\{q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{a}} > 0\}} + R^{\mathfrak{a}}(x; .)^2 \mathbf{1}_{\{q^{\mathfrak{a}} - \Delta\alpha^{\mathfrak{a}} \leq 0\}}
\end{cases}
$$

- When there is no more remaining limit orders, the market maker can place limit orders again. In this case, we have $\Delta\zeta \ = (0, 0, \Delta\ell^{\mathfrak{b}} \mathbf{1}_{\{b^{\mathfrak{b}} = 0\}}, \Delta\ell^{\mathfrak{a}} \mathbf{1}_{\{b^{\mathfrak{a}} = 0\}}, 0, 0)$

$$
\begin{cases}
g' = g \\
i' = i \\
(n^{\mathfrak{b}})' = (q^{\mathfrak{b}} + 1) \mathbf{1}_{\{b^{\mathfrak{b}} = 0\}} + n^{\mathfrak{b}} \mathbf{1}_{\{b^{\mathfrak{b}} \neq 0\}} \\
(n^{\mathfrak{a}})' = (q^{\mathfrak{a}} + 1) \mathbf{1}_{\{b^{\mathfrak{a}} = 0\}} + n^{\mathfrak{a}} \mathbf{1}_{\{b^{\mathfrak{a}} \neq 0\}} \\
(b^{\mathfrak{b}})' = \Delta\ell^{\mathfrak{b}} \mathbf{1}_{\{b^{\mathfrak{b}} = 0\}} + b^{\mathfrak{b}} \mathbf{1}_{\{b^{\mathfrak{b}} \neq 0\}} \\
(b^{\mathfrak{a}})' = \Delta\ell^{\mathfrak{b}} \mathbf{1}_{\{b^{\mathfrak{a}} = 0\}} + b^{\mathfrak{a}} \mathbf{1}_{\{b^{\mathfrak{a}} \neq 0\}} \\
(p^{\mathfrak{b}})' = p^{\mathfrak{b}} \\
(p^{\mathfrak{a}})' = p^{\mathfrak{a}} \\
(q^{\mathfrak{b}})' = q^{\mathfrak{b}} + \Delta\ell^{\mathfrak{b}} \mathbf{1}_{\{b^{\mathfrak{b}} = 0\}} \\
(q^{\mathfrak{a}})' = q^{\mathfrak{a}} + \Delta\ell^{\mathfrak{b}} \mathbf{1}_{\{b^{\mathfrak{a}} = 0\}}
\end{cases}
$$

- The market maker choose to place limit order within the spread on the bid side when the spread is two tick sizes. We have $\Delta\zeta \ = (0, 0, 0, 0, \Delta\ell^{\mathfrak{b}, \frac{1}{2}} \mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} = 2\delta\}}, 0)$

$$
\begin{cases}
g' = g \\
i' = i \\
(n^{\mathfrak{b}})' = 1 \mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} = 2\delta\}} + n^{\mathfrak{b}} \mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} = \delta\}} \\
(n^{\mathfrak{a}})' = n^{\mathfrak{a}} \\
(b^{\mathfrak{b}})' = \Delta\ell^{\mathfrak{b}, \frac{1}{2}} \mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} = 2\delta\}} + b^{\mathfrak{b}} \mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} = \delta\}} \\
(b^{\mathfrak{a}})' = b^{\mathfrak{a}} \\
(p^{\mathfrak{b}})' = (p^{\mathfrak{b}} + \delta) \mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} = 2\delta\}} + p^{\mathfrak{b}} \mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} = \delta\}} \\
(p^{\mathfrak{a}})' = p^{\mathfrak{a}} \\
(q^{\mathfrak{b}})' = \Delta\ell^{\mathfrak{b}, \frac{1}{2}} \mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} = 2\delta\}} + q^{\mathfrak{b}} \mathbf{1}_{\{p^{\mathfrak{a}} - p^{\mathfrak{b}} = \delta\}} \\
(q^{\mathfrak{a}})' = q^{\mathfrak{a}}
\end{cases}
$$

- The market maker choose to place limit order within the spread on the ask side when the spread is two tick sizes. We have $\Delta\zeta = (0,0,0,0,0,\Delta\ell^{\mathfrak{a},\frac{1}{2}}\mathbf{1}_{\{p^{\mathfrak{a}}-p^{\mathfrak{b}}=2\delta\}})$

$$\begin{cases} g' = g \\ i' = i \\ n'^{\mathfrak{b}} = n^{\mathfrak{b}} \\ n'^{\mathfrak{a}} = 1\mathbf{1}_{\{p^{\mathfrak{a}}-p^{\mathfrak{b}}=2\delta\}} + n^{\mathfrak{a}}\mathbf{1}_{\{p^{\mathfrak{a}}-p^{\mathfrak{b}}=\delta\}} \\ b'^{\mathfrak{b}} = b^{\mathfrak{b}} \\ b'^{\mathfrak{a}} = \Delta\ell^{\mathfrak{a},\frac{1}{2}}\mathbf{1}_{\{p^{\mathfrak{a}}-p^{\mathfrak{b}}=2\delta\}} + b^{\mathfrak{a}}\mathbf{1}_{\{p^{\mathfrak{a}}-p^{\mathfrak{b}}=\delta\}} \\ p'^{\mathfrak{b}} = p^{\mathfrak{b}} \\ p'^{\mathfrak{a}} = (p^{\mathfrak{a}}-\delta)\mathbf{1}_{\{p^{\mathfrak{a}}-p^{\mathfrak{b}}=2\delta\}} + p^{\mathfrak{a}}\mathbf{1}_{\{p^{\mathfrak{a}}-p^{\mathfrak{b}}=\delta\}} \\ q'^{\mathfrak{b}} = q^{\mathfrak{b}} \\ q'^{\mathfrak{a}} = \Delta\ell^{\mathfrak{a},\frac{1}{2}}\mathbf{1}_{\{p^{\mathfrak{a}}-p^{\mathfrak{b}}=2\delta\}} + q^{\mathfrak{a}}\mathbf{1}_{\{p^{\mathfrak{a}}-p^{\mathfrak{b}}=\delta\}} \end{cases}$$

4 The optimal control problem

4.0.1 Value function and dynamic programming principle

The aim of the controller is, for $(t,x) \in [0,T]\times\mathbf{X}$, to maximize the expected utility $\mathbb{E}[U(X_T^{t,x})]$, in which

$$U(x) := -\exp\left(-\eta\{g + i^+p^{\mathfrak{b}} - i^-p^{\mathfrak{a}} - \kappa([i^+ - q^{\mathfrak{b}}]^+ + [i^- - q^{\mathfrak{a}}]^+)\}\right) \tag{VIII.2}$$

for some $\eta, \kappa > 0$. In the above, η is the risk aversion parameter. The quantity $i^+p^{\mathfrak{b}} - i^-p^{\mathfrak{a}}$ corresponds to the liquidation value of the inventory if the bid and ask queues are big enough to absorb it. The expression starting from κ is a penalty term that takes into account the number of shares that will not be liquidated at the best limit.

We can now define the value function:

$$\mathrm{v}(t,x) := \sup_{\phi\in\Phi^{t,x}} J(t,x;\phi) \quad \text{where } J(t,x;\phi) := \mathbb{E}[U(X_T^{t,x,\zeta})].$$

Remark 4.1. *Looking at a trivial control with $\tau_1 > T$ $\mathbb{P}$-a.s. highlights that $-\exp(-\eta g) \leq \mathrm{v} \leq 0$.*

At this stage, one cannot assume any regularity on v.

4.0.2 Characterization as a viscosity solution

The derivation of the PDE relies on a dynamic programming principle.

We shall denote, for $\mathbf{a} \in \mathbf{A}$ and $(t,x) \in [0,T]\times\mathbf{X}$, $\mathbb{P}_X(\cdot \mid t,x,\mathbf{a})$ the probability distribution for $X_{t+}^{t,x}$ after the jump of the control played at time $\{\tau_i = t\}$. Recall that this randomness comes from the fact that a Queue can be depleted, and in this case, the regeneration, at the price or with a price move, is random.

We define the operator $\mathcal{K}$, for a bounded measurable function φ and for $(t,x) \in [0,T] \times \mathbf{X}$, by

$$\mathcal{K}^{\mathbf{a}}\varphi(t,x) := \int \varphi(t,x') d\mathbb{P}_X(x' \mid t,x,\mathbf{a});$$

and $\mathcal{K}$ the operator

$$\mathcal{K}\varphi(t,x) := \sup_{\mathbf{a}\in\mathbf{A}} \mathcal{K}^{\mathbf{a}}\varphi(t,x).$$

We also introduce the Dynkin operator $\mathcal{L}$ associated to the diffusion of X. The aim is to show that, with a comparison theorem, v is the unique viscosity solution of

$$\begin{aligned} \min\{-\mathcal{L}\varphi, \varphi - \mathcal{K}\varphi\} = 0 \text{ on } [0,T) \times \mathbf{X} \\ \varphi - u = 0 \text{ on } \{T\} \times \mathbf{X} \end{aligned} \tag{VIII.3}$$

where the notion of *viscosity solution* of the PDE above is taken in the sense of [9, Definition 3.1.].

Proposition 4.2. *Fix $(t,x) \in [0,T] \times \mathbf{X}$, and let be θ the first exit from a Borel set $B \subset [0,T] \times \mathbf{X}$ containing (t,x). Then,*

$$\mathrm{v}(t,x) \le \sup_{\phi\in\Phi^{t,x}} \mathbb{E}\left[\mathrm{v}^*(\theta, X_\theta^{t,x,\phi})\mathbf{1}_{\{\theta<\tau_1\}} + \mathcal{K}^{\mathbf{a}_1}\mathrm{v}^*(\tau_1^\phi, X_{\tau_1}^{t,x,\phi})\mathbf{1}_{\{\theta\ge\tau_1\}}\right],$$

Proof. The proof is given in [9] in a Brownian framework, in our case with a finite random Poisson measure, we refer to it: the arguments are the same. Note that we do not have their $\mathcal{K}_T$ operator since there is no randomness at the final date T.

From this, we get:

Proposition 4.3. *The function* v *is a subsolution of (VIII.3).*

Proof. See [9], the proof is given in a Brownian framework, the finite jump process, in which the jumps are bounded, and only need to take care of the choice of the ball.

For the super-solution property, [9] gives the following discrete version.

Proposition 4.4. *Let $\Phi_n^{t,x} \subset \Phi^{t,x}$ the set of admissible controls in which, for all $i \ge 1$, on $\{\tau_i \le T\}$, we have $\tau_i \in \pi_n$ $\mathbb{P}$-a.s. where $\pi_n := \{t\} \cup \frac{T}{2^n}\{0,\dots,2^n\}$. Define v_n as the value function where the supremum among the controls is taken over $\Phi_n^{t,x}$. Then, for any $\mathcal{F}$-stopping time θ^ϕ,*

$$\mathrm{v}_n(t,x) = \sup_{\phi\in\Phi_n^{t,x}} \mathbb{E}\left[\mathrm{v}_n(\theta^\phi, X_{\theta^\phi}^{t,x})\right]$$

Proof. Again, see [9].

Now, Assume that we have a comparison principle, then, again from [9], we have the super solution property.

Proposition 4.5. *Assume that a comparison theorem hold for (VIII.3). The function* v *is a supersolution of (VIII.3).*

However, for the result above and the numerical scheme, we need to have a comparison theorem.

Theorem 4.1. *Assume that* v^* *is a subsolution of (VIII.3) and that* v_* *is a supersolution of (VIII.3), and that* $\mathrm{v}^*(T,\cdot) \leq \mathrm{v}_*(T,\cdot)$. *Then,*

$$\mathrm{v}^* \leq \mathrm{v}_* \;\; on \; [0,T] \times \mathbf{X}.$$

Proof. The proof, as usual, relies on the build of a strict super solution. The paper [9] provides the result if we can find a function Ψ which satisfies their conditions. The function

$$\begin{aligned} \Psi : \mathbf{X} &\to \mathbb{N} \\ x &\mapsto I^* - i \end{aligned}$$

fulfills them.

4.0.3 Dimension reduction

For $\eta > 0$ and $\kappa = 2\delta$, starting at initial data $(t,x) = (t, g, p^{\mathfrak{b}}, p^{\mathfrak{a}}, i, q^{\mathfrak{b}}, q^{\mathfrak{a}}, n^{\mathfrak{b}}, n^{\mathfrak{a}}, b^{\mathfrak{b}}, b^{\mathfrak{a}})$, and the utility function can be rewriten, with $s := p^{\mathfrak{b}} - p^{\mathfrak{a}}$,

$$U_\eta(x) = -\exp\left[-\eta\left(g + ip^{\mathfrak{b}} - i^- s - \left[\delta(i^+ - q^b)^+ + (s+\delta)(i^- - q^a)^+\right]\right)\right]$$

Since for $s \geq t$, we have $G_s^{t,g} = g + G_s^{t,0}$, we first have,

$$\begin{aligned} \mathrm{v}(t,x) &= e^{-\eta g} \sup_{\phi \in \Phi_{t,x}} \mathbb{E}\left[U\left(G_T^{t,0,p}, P_T^{\mathfrak{b},t,p}, S_T^{t,s}, I_T^{t,i}, Q_T^{t,q^b}, Q_T^{t,q^a}, N_T^{t,n^b}, N_T^{t,n^a}, B_T^{t,b^b}, B_T^{t,b^a}\right)\right] \\ &= e^{-\eta g}\mathrm{v}(t, 0, p^{\mathfrak{b}}, s, i, q^b, q^a, n^b, n^a, b^b, b^a) \end{aligned}$$

Then, an optimal or sequence of $\epsilon-$optimal for $g \in \mathbb{R}$ is also one for $g = 0$.

For the next reduction, we shall assume that:

Assumption 1. *For all* $p^{\mathfrak{b}}, p^{\mathfrak{a}} \in \delta\mathbb{Z}$ *such that* $0 \leq p^{\mathfrak{b}} - p^{\mathfrak{a}} \leq 2\delta$, *the measure of jumps* ν *depends only on* $s = p^{\mathfrak{b}} - p^{\mathfrak{a}}$, *i.e. there is a measure* ν' *such that*

$$\nu(p^{\mathfrak{b}}, p^{\mathfrak{a}}, q^{\mathfrak{b}}, q^{\mathfrak{a}}) = \nu'(s, q^{\mathfrak{b}}, q^{\mathfrak{a}}).$$

Since the measure ν in independent of the level of the price (only the tick), we also have $P_s^{t,p} = p + P_s^{t,0}$. Moreover, if for a control $\phi \in \Phi_{t,x}$, τ_i^ϕ are the stopping times at which we sent orders.

We denote by $(\tau_i^\phi)_{i\geq 1}$ the stopping time associated to ϕ and $(a_i^{\mathfrak{b}})_{i\geq 1}$ and $(a_i^{\mathfrak{a}})$ the associated quantity for aggressive orders at the bid and the ask. We denote by $\vartheta_i^{\mathfrak{b},\phi}$ (resp. $\vartheta_i^{\mathfrak{a},\phi}$) the execution times of limit orders to buy new (resp. sell) shares of size $\ell_i^{\mathfrak{b},\phi}$ (resp. $\ell_i^{\mathfrak{b},\phi}$). Then,

$$\begin{aligned}
G_T^{t,0,p} &= -\sum_{i\geq 1} a_i^{\mathfrak{a}}(P_{\tau_i^\phi}^{t,p} + S_{\tau i^\phi}^{t,s})\mathbf{1}_{\{\tau_i^\phi \leq T\}} + \sum_{i\geq 1} a_i^{\mathfrak{b}} P_{\tau_i^\phi}^{t,p}\mathbf{1}_{\{\tau_i^\phi \leq T\}} - \sum_{i\geq 1} \ell_i^{\mathfrak{b}} P_{\vartheta_i^\phi}^{t,p}\mathbf{1}_{\{\vartheta_i^\phi \leq T\}} + \sum_{i\geq 1} \ell_i^{\mathfrak{a}}(P_{\vartheta_i^\phi}^{t,p} + S_{\vartheta_i^\phi}^{t,s})\mathbf{1}_{\{\vartheta_i^\phi \leq T\}} \\
&= p\left(-\sum_{i\geq 1} a_i^{\mathfrak{a}}\mathbf{1}_{\{\tau_i^\phi \leq T\}} + \sum_{i\geq 1} a_i^{\mathfrak{b}}\mathbf{1}_{\{\tau_i^\phi \leq T\}} - \sum_{i\geq 1} \ell_i^{\mathfrak{b}}\mathbf{1}_{\{\vartheta_i^\phi \leq T\}} + \sum_{i\geq 1} \ell_i^{\mathfrak{a}}\mathbf{1}_{\{\vartheta_i^\phi \leq T\}} \right) \\
&\quad - \sum_{i\geq 1} a_i^{\mathfrak{a}}(P_{\tau_i^\phi}^{t,0} + S_{\tau i^\phi}^{t,s})\mathbf{1}_{\{\tau_i^\phi \leq T\}} + \sum_{i\geq 1} a_i^{\mathfrak{b}} P_{\tau_i^\phi}^{t,0}\mathbf{1}_{\{\tau_i^\phi \leq T\}} - \sum_{i\geq 1} \ell_i^{\mathfrak{b}} P_{\vartheta_i^\phi}^{t,0}\mathbf{1}_{\{\vartheta_i^\phi \leq T\}} + \sum_{i\geq 1} \ell_i^{\mathfrak{a}}(P_{\vartheta_i^\phi}^{t,0} + S_{\vartheta_i^\phi}^{t,s})\mathbf{1}_{\{\vartheta_i^\phi \leq T\}}
\end{aligned}$$

Moreover, we can see that

$$I_T^{t,i} = i + \sum_{i\geq 1} a_i^{\mathfrak{a}}\mathbf{1}_{\{\tau_i^\phi \leq T\}} - \sum_{i\geq 1} a_i^{\mathfrak{b}}\mathbf{1}_{\{\tau_i^b \leq T\}} + \sum_{i\geq 1} \ell_i^{\mathfrak{b}}\mathbf{1}_{\{\vartheta_i^\phi \leq T\}} - \sum_{i\geq 1} \ell_i^{\mathfrak{a}}\mathbf{1}_{\{\vartheta_i^\ell \leq T\}}$$

Then

$$G_T^{t,0,p} = p(i - I_T^{t,i}) + G_T^{t,0,0}.$$

We then have

$$\begin{aligned}
\mathrm{v}(t,x) &= e^{-\eta(g+pi)} \sup_{\phi\in\Phi_{t,x}} \mathbb{E}\left[U\left(G_T^{t,0,0}, P_T^{t,0}, S_T^{t,s}, I_T^{t,i}, Q_T^{t,q^b}, Q_T^{t,q^a}, N_T^{t,n^b}, N_T^{t,n^a}, B_T^{t,b^b}, B_T^{t,b^a}\right)\right] \\
&= e^{-\eta(g+pi)}\mathrm{v}(t,0,0,s,i,q^b,q^a,n^b,n^a,b^b,b^a)
\end{aligned}$$

Then, an optimal or sequence of $\epsilon-$optimal for $(g,p^{\mathfrak{b}}) \in (\delta\mathbb{Z})^2$ is also one for $(g,p) = (0,0)$.

Moreover, we can use a symmetric effect which implies that $\mathrm{v}(t,g,p,i,s,q^b,q^a,n^b,n^a,b^b,b^a) = \mathrm{v}(t,g,p+s,-i,s,q^a,q^b,n^a,n^b,b^a,b^b)$. This comes from the fact that the random Poisson measure is symmetric through bid and ask and the final criteria is also symmetric in the same sens when the mid price is 0, so when $p^{\mathfrak{a}} = -p^{\mathfrak{b}}$.

We deduce that

$$\begin{aligned}
\mathrm{v}(t,0,0,i,s,q^b,q^a,n^b,n^a,b^b,b^a) &= \mathrm{v}(t,0,s,-i,s,q^a,q^b,n^a,n^b,b^a,b^b) \\
&= e^{-\eta si}\mathrm{v}(t,0,0,-i,s,q^a,q^b,n^a,n^b,b^a,b^b).
\end{aligned}$$

5 Numerical resolution

There is only the time variable which evolves continuously, we introduce for $h > 0$ and a bounded measurable function φ,

$$\Delta_t^h\varphi(t,x) = h^{-1}(\varphi(t+h,x) - \varphi(t,x)).$$

We also introduce

$$I_t^h\varphi(t,x) = \lambda(p,q)h\int \left[\varphi(t+h,x+\beta(t,x,u)) - \varphi(t,x)\right] d\mu(p,q,u),$$

in which μ is the distribution of the orders when one occur, associated to ν and λ is its finite intensity. Note that, since the state space for μ is finite, then I_t^h can be computed exactly. We introduce:

$$\mathcal{L}^h\varphi(t,x) := \Delta_t^h\varphi(t,x) + I_t^h\varphi(t,x).$$

Now, since we have the dimension reduction, recall that, for $(g,p) \in (\delta\mathbb{Z})^2$,

$$\mathrm{v}(t,g,p^{\mathfrak{b}},s,i,\cdot) = e^{-\eta(g+ip^{\mathfrak{b}})}\mathrm{v}(t,0,0,s,i,\cdot); \tag{VIII.4}$$

we introduce

$$\mathcal{P}\varphi(t,x) = e^{-\eta(g+ip^{\mathfrak{b}})}\varphi(t,0,0,\cdot).$$

We also approximate the operator $\mathcal{K}\varphi$ with:

$$\mathcal{K}^h\varphi(t,x) := \sup_{\mathbf{a}\in\mathbf{A}} \int \mathcal{P}\varphi(t+h,x')d\mathbb{P}_X(x' \mid t,x,\mathbf{a}).$$

Again, for each $\mathbf{a} \in \mathbf{A}$, it can be computed exactly since the number of events is finite and if $\mathbf{A}$ is finite, this reduces to compute it for each $\mathbf{a}$ and to choose the optimal one.

Our numerical scheme consists in solving, with $\mathbf{T}^h := [0,T] \cap h\mathbb{N}$ for $h > 0$ such that $T/h \in \mathbb{N}$ and with $\mathbf{X}' := \{x \in \mathbf{X} : g = p^{\mathfrak{b}} = 0\}$,

$$\begin{aligned} \min\left\{-\mathcal{L}^h\varphi, \varphi - \mathcal{K}^h\varphi\right\} = 0 \text{ on } \mathbf{T}^h\backslash\{T\} \times \mathbf{X}' \\ \varphi - u = 0 \text{ on } \{T\} \times \mathbf{X}' \end{aligned} \tag{VIII.5}$$

Remark 5.1. *We do note introduce some h_1 nor bound for the state space grid since this one discrete and finite.*

Proposition 5.2. *Let v^h denotes the solution of (VIII.5) extended to $\mathbf{T}^h \times \mathbf{X}$ using (VIII.4). Then, $\mathrm{v}^h \to \mathrm{v}$ when $h \to 0$.*

Proof. Since u is bounded and we have the comparison, one easily checks that our scheme satisfies the conditions of [10, Theorem 2.1.].

Chapter IX

Optimal high frequency strategy and VWAP strategy

In this chapter, for the reader's convenience, we give the detailed description of the HF and IB's strategy which are well established pair-trading and VAWP strategies.

1 An optimal high frequency strategy: Pair trading and impulse control

We present a HFT strategy also within the stochastic optimal control framework. We consider a pair trading strategy where the trader invests in the difference of two highly correlated assets. Here, we choose the futures price of the stock to be the second asset.

1.1 Impulse control on the dynamics of the spread

Suppose that we have a pair of financial assets that are highly correlated and we try to play on the return of the difference[1] of these two assets. Let F be the futures price of an asset S_1. We denote $S = F - S_1$ and assume that this difference evolves according Ornstein-Uhlenbeck process. That is,

$$dS_t = \rho(b - S_t)dt + \sigma dW_t,$$

where ρ is the strength of mean reversion, b is the average of mean reversion and σ is the volatility of the process. This means that if S_t is very low, then the difference has the tendency to return to the value b, and vice versa.

We model the trading strategy as follows. Either we sell the spread when we expect that it is going to decrease and we reverse our position later, or we buy the spread and resell it afterwards if we expect that it is going to increase. Moreover, we assume that each purchase/sale is subject to a fixed cost, $c > 0$. Therefore, it is not possible to buy/sell continuously. Our goal is to find the right time to trade. To achieve this, we begin by modeling this strategy as an impulse control problem.

An impulse control problem is an adapted process ϕ of the form

$$\phi_t = \sum_{i \geq 1} \xi_i \mathbf{1}_{t \in [\tau, \tau_{i+1})},$$

where $(\tau_i)_{i \geq 1}$ is an increasing sequence of stopping times and $(\xi)_{i \geq 1}$ is a sequence of random variables, with values in $\{-1, 1\}$. The times τ_i will be our purchase or sale times. If $\xi = 1$, it

[1] We also call this difference as spread. This explains the title of this subsection.

represents a purchase, otherwise it is a sale. The position at time t is thus given by

$$N_t = \sum_{i\geq 1} \xi_i \mathbf{1}_{\tau_i \leq t}.$$

We introduce the process

$$V_t = V_0 - \sum_{t\geq 1} (\xi_i S_{\tau_i} + c)\mathbf{1}_{\tau_i \leq t}.$$

Hence, $V_t + S_t N_t - c|N_t|$ is the value of the trading gains if we liquidate the position at time t.

Thereafter, we denote $(S^{t,s}, V^{t,s,x,\phi}, N^{t,n,\phi})$ the processes starting at time t, with initial conditions $S_t^{t,s} = s$, $V_t^{t,s,x,\phi} = x$ and $N_t^{t,n,\phi} = n$. We say that a control ϕ is admissible from t and from a position $n \in \{-1, 0, 1\}$, if $N^{t,n,\phi}$ take values in $\{-1, 0, 1\}$ at all previous times. In other words, we only allow ourselves to hold at most a spread unit at each instant. We denote by $\mathcal{A}(t, n)$ the set off these admissible controls. Given a utility function U, we consider the expected gain

$$J(t, s, x, n; \phi) := \mathbb{E}[U(V_T^{t,s,x,\phi} + N_T^{t,n,\phi} S_T^{t,s} - c|N_T^{t,n,\phi}|)].$$

The value function of our problem is

$$v(t, s, x, n) = \sup_{\phi\in\mathcal{A}(t,n)} J(t, s, x, n; \phi)$$

1.2 Dynamic programming principle and quasi-variational Hamilton-Jacobi-Belman equation

Theorem 1.1. *Let $(t, s, x, n) \in D := [0, T] \times \mathbb{R} \times \mathbb{R} \times \{1, 0, 1\}$ and $\theta \in \mathcal{T}_t$. Then*

$$v(t, s, x, n) = \sup_{\phi\in\mathcal{A}(t,n)} \mathbb{E}[v(\theta, S_\theta^{t,s}, V_\theta^{t,s,x,\phi}, N_\theta^{t,n,\phi})].$$

In particular, taking $\theta = t$, and considering a control that acts immediately, we obtain

$$v(t, s, x, n) \geq \max_{a\in A(n)} v(t, s, x - as - c, n + a),$$

where

$$A(n) = \{a \in \{-1, 1\} : n + a \in \{-1, 0, 1\}\}.$$

On the other hand, we consider a control that consists in never buy/sell in the interval $[t, \theta]$, we have

$$v(t, s, x, n) \geq \mathbb{E}[v(\theta, S_\theta^{t,s}, x, n)].$$

By applying Itô's lemma and considering a sequence of stopping times approaching t, we obtain

$$0 \geq \mathcal{L}v(t, s, x, n) := v_t(t, s, x, n) + \rho(b - s)v_s(t, s, x, n) + \frac{1}{2}\sigma^2 v_{ss}(t, s, x, n).$$

If v is regular enough, it verifies

$$0 \geq \max\{\mathcal{L}v(t, s, x, n), \max_{a\in A(n)} v(t, s, x - as - c, n + a) - v(t, s, x, n)\}.$$

As it is stated in Theorem 1.2, we can show that the above inequality can not be strict.

Theorem 1.2. *Let $v \in C^{1,2,0,0}([0,T)\times\mathbb{R}\times\mathbb{R}\times\{-1,0,1\})$ and continuous on D, then it verifies*

$$0 = \max\{\mathcal{L}v(t,s,x,n), \max_{a\in A(n)} v(t,s,x-as-c,n+a) - v(t,s,x,n)\}. \tag{IX.1}$$

on $[0,T)\times\mathbb{R}\times\mathbb{R}\times\{-1.0,1\}$. Moreover, $v(T,s,x,n) = U(x+sn-c|n|)$ on $\mathbb{R}\times\mathbb{R}\{-1,0,1\}$.

1.3 Derivation of the optimal control and Numerical solution

To obtain the optimal strategy associated with (IX.1), we assume that w is a regular solution. Moreover, we suppose that we start from time 0 with initial conditions s,x,n given, which are omitted later in the associated processes. Consider the following sequence of stopping times and actions starting from time 0:

$$\begin{aligned}
\hat{\tau}_{i+1} &:= \inf\left\{t > \hat{\tau}_i : w(t,S_t,V_t^{\hat{\phi}},N_t^{\hat{\phi}}) = \max_{a\in A(N_t^{\hat{\phi}})} w(t,S_t,V_t^{\hat{\phi}} - aS_t - c, N_t^{\hat{\phi}} + a)\right\} \\
\hat{\xi}_{i+1} &:= \arg\max\left\{w(\hat{\tau}_{i+1}, S_{\hat{\tau}_{i+1}}, V_{\hat{\tau}_{i+1}}^{\hat{\phi}} - aS_{\hat{\tau}_{i+1}} - c, N_{\hat{\tau}_{i+1}}^{\hat{\phi}} + a), a \in A(N_{\hat{\tau}_{i+1}}^{\hat{\phi}})\right\},
\end{aligned}$$

with $\hat{\tau}_0 := 0$ and $\hat{\phi}$ given by $(\hat{\tau}_i, \hat{\xi}_i)_{i\geq 1}$. Since w verifies (IX.1), we obtain

$$\begin{aligned}
&U(V_T^{\hat{\phi}} + N_T^{\hat{\phi}} S_T - c|N_T^{\hat{\phi}}|) \\
&\quad = w(T,S_T,V_T^{\hat{\phi}},N_T^{\hat{\phi}}) \\
&\quad = w(0,s,x,n) + \sum_{i\geq 0}\int_{\hat{\tau}_i\wedge T}^{\hat{\tau}_{i+1}\wedge T} \mathcal{L}w(t,S_t,V_t^{\hat{\phi}},N_t^{\hat{\phi}})dt \\
&\quad + \sum_{i\geq 0}\int_{\hat{\tau}_i\wedge T}^{\hat{\tau}_{i+1}\wedge T} w_s(t,S_t,V_t^{\hat{\phi}},N_t^{\hat{\phi}})\sigma dW_t \\
&\quad + \sum_{i\geq 1}\mathbf{1}_{\hat{\tau}_i\geq T}\left(w(t,S_t,V_t^{\hat{\phi}} - \hat{\xi}_i S_{\hat{\tau}_i} - c, N_t^{\hat{\phi}} + \hat{\xi}_i) - w(t,S_t,V_t^{\hat{\phi}},N_t^{\hat{\phi}})\right) \\
&\quad = w(0,s,x,n) + \int_0^T w_s(t,S_t,V_t^{\hat{\phi}},N_T^{\hat{\phi}})\sigma dW_t
\end{aligned}$$

which implies that

$$v(0,s,x,n) \geq \mathbb{E}[U(V_T^{\hat{\phi}} + N_T^{\hat{\phi}} S_T - |N_T^{\hat{\phi}}|)] = w(0,s,x,n)$$

Moreover, if $\hat{\phi}$ is another control associated with the sequence $(\tau_i,\xi_i)_{i\geq 1}$, then the fact that w verify (IX.1) implies

$$
\begin{aligned}
&U(V_T^\phi + N_T^\phi S_T - c|N_T^\phi|)\\
&\quad = w(T, S_T, V_T^\phi, N_T^\phi)\\
&\quad = w(0, s, x, n) + \sum_{i\geq 0} \int_{\tau_i \wedge T}^{\tau_{i+1}\wedge T} \mathcal{L}w(t, S_t, V_t^\phi, N_t^\phi)dt\\
&\quad + \sum_{i\geq 0} \int_{\tau_i \wedge T}^{\tau_{i+1}\wedge T} w_s(t, S_t, V_t^\phi, N_t^\phi)\sigma dW_t\\
&\quad + \sum_{i\geq 1} \mathbf{1}_{\tau_i \geq T} \Big(w(t, S_t, V_t^\phi - \xi_i S_{\tau_i} - c, N_t^\phi + \xi_i) - w(t, S_t, V_t^\phi, N_t^\phi)\Big)\\
&\quad \leq w(0, s, x, n) + \int_0^T w_s(t, S_t, V_t^\phi, N_T^\phi)\sigma dW_t.
\end{aligned}
$$

Thus, using the expected value, we deduce

$$J(0, s, x, n) \leq w(0, s, x, n).$$

Since this is true for all $\phi \in (0, n)$, we deduce that $v(0, s, x, n) \leq w(0, s, x, n)$. Therefore, $w = v$ and $\hat{\phi}$ is the optimal strategy.

Theorem 1.3. *Let w satisfy the previous assumptions. Moreover, suppose that w is bounded. Then, $w(0, s, x, n) = v(0, s, x, n)$ and the optimal strategy is given by $\hat{\phi}$ given above.*

The optimal strategy is actually very simple. It says that we can only act at times t such that

$$v(t, S_t, V_t^{\hat{\phi}}, N_t^{\hat{\phi}}) = \max_{a\in A(N_t^{\hat{\phi}})} v(t, S_t, V_t^{\hat{\phi}} - aS_t - c, N_t^{\hat{\phi}} + a).$$

Let us consider a grid of points in time $\{t_\ell, \ell \leq L\}$ with $t_\ell = \ell T/L$, and in space $\{s_k, k \leq K\} \times \{x_m, m \leq M\}$ with $s_k = -\bar{s}_0 + 2k\bar{s}_0/K$ and $x_m = -\bar{x}_0 + 2m\bar{x}_0/M$ where $L, K, M \in \mathbb{N}$ and $\bar{s}_0, \bar{x}_0 > 0$ are given. The time derivative is approximated as

$$\Delta_t^L v(t, s, x, n) := \frac{v(t + L^{-1}, s, x, n) - v(t, s, x, n)}{L^{-1}}.$$

For the term of the first derivative in space, we use the "upwind" schema:

$$\Delta_s^{L,K} v(t, s, x, n) := \begin{cases} \frac{v(t+L^{-1}, s+K^{-1}, x, n) - v(t, s, x, n)}{K^{-1}} & \text{if } b - s \geq 0 \\ \frac{v(t, s, x, n) - v(t+L^{-1}, s+K^{-1}, x, n)}{K^{-1}} & \text{if } b - s < 0 \end{cases}$$

Finally, the second derivative is approximated gy

$$
\begin{aligned}
&\Delta_{ss}^{L,K} v(t, s, x, n)\\
&\qquad := \frac{v(t + L^{-1}, s + K^{-1}, x, n) + v(t + L^{-1}, s - K^{-1}, x, n) - 2v(t, s, x, n)}{K^{-2}}
\end{aligned}
$$

This gives the following numerical scheme:

(i) For each $k \leq K$, $m \leq M$ and $n \in \{-1, 0, 1\}$, we set $\varphi_{K,L,M}(T, s_k, x_m, n) = U(x_m + s_k - c|n|)$.

(ii) Given $\varphi_{K,L,M}(t_{\ell+1}, \cdot)$, we set $\tilde{\varphi}_{K,L,M}(t_{\ell+1}, \cdot) := \varphi_{K,L,M}(t_{\ell+1}, \cdot)$ and for each $k \leq K$, $m \leq M$ and $n \in \{-1, 0, 1\}$, we calculate the solution $\tilde{\varphi}_{K,L,M}(t_\ell, s_k, x_m, n)$ of

$$0 = \Delta_t^L \tilde{\varphi}_{K,L,M}(t_\ell, s_k, x_m, n) + \rho(b - s_k)\Delta_s^{L,K} \tilde{\varphi}_{K,L,M}(t_\ell, s_k, x_m, n) + \frac{1}{2}\sigma^2 \Delta_{ss}^{L,K} \tilde{\varphi}_{K,L,M}(t_\ell, s_k, x_m, n)$$

and we define $\varphi(t_\ell, s_k, x_m, n)$ as equal to

$$\max\{\tilde{\varphi}_{K,L,M}(t_\ell, s_k, x_m, n), \max_{a \in A(n)} \varphi_{K,L,M}(t_\ell + L^{-1}, s_k, x_m - as_k - c, n + a)\}$$

Moreover, we need boundary conditions for the points s_0, s_K, x_0 and x_M. For, this, one can take for example the value of the utility associated to U, which corresponds to say that one stops trading when one reaches these boundary points. This scheme is convergent, as long as the sequence of functions computed remains locally bounded (for instance if U is bounded), since $K, L, M \to \infty$ with $K^2/L \to 0$ and $\bar{s}_0, \bar{x}_0 \to \infty$.

Remark 1.1. *If one takes $U(z) = -e^{-\eta z}$, $\eta > 0$, then*

$$v(t, s, x, n) = e^{-\eta x} v(t, s, 0, n) =: e^{-\eta x} \tilde{v}(t, s, n)$$

and $\tilde{v}$ verifies

$$0 = \max\{\mathcal{L}\tilde{v}(t, s, n), \max_{a \in A(n)} \{e^{\eta(as+c)} \tilde{v}(t, s, n + a) - \tilde{v}(t, s, n)\}\},$$

in $[0, T) \times \mathbb{R} \times \mathbb{R}\{-1, 0, 1\}$ and $\tilde{v}(T, s, n) = U(sn - c|n|)$ in $\mathbb{R} \times \mathbb{R} \times \{-1, 0, 1\}$. Thus, we reduce one dimension in our problem, which simplifies considerably the scheme of our numerical solution.

2 A VWAP based strategy for portfolio liquidation

Here, we present a VWAP (Volume Weighted Average Price) based trading strategy frequently practiced by institutional brokers. In fact, the VWAP strategies are usually used by market practitioners when it comes to execution of large orders. These strategies can be used either to buy or to sell shares of financial assets. In this section, we present a VWAP selling strategy, but it is adaptable to a buy one with ease. In practice, traders use the VWAP as a benchmark to ensure that their purchases are in line with the available volume in the market. Hence, these strategies reduce the impact on the natural market dynamics of an asset price.
We present here a VWAP based strategy in the Almgreen-Chriss[2] framework. First, we consider the continuous time case. Next, we present the discrete time framework (i.e consistent with our discrete time approximation in Sections VIII and 1). Such strategies are inspired from [7]

[2]See [7]. This seminal work combines both the expected cost of execution and the risk that the price would move more over the course of the execution process.

and where extended by, for instance, Guéant (see [24]). Details of the strategy that we present here can be found in [24, Section 4.4].

2.1 Discrete time framework - VWAP based strategy

As in Section 2.2, we regard $T < \infty$ as the final time horizon and $Q_0 > 0$ the quantity to sell. We consider a regular temporal grid with time step Δ and let $n \in \{0, \cdots, N-1\}$ the time index, with $N = [T/\Delta]$. We split the quantity Q_0 in N slices, v_n, such that $\sum_{n=0}^{N-1} v_n = q_0$. To fulfill its goal, the agent controls the quantity v_n to sell at every time step. The price dynamic satisfy

$$S_{n+1} = S_n + \beta v_n \Delta + \sigma_n \sqrt{\Delta} \epsilon_{n+1}, \qquad \forall n \in \{0, \cdots, N-2\}.$$

The agent wealth satisfy

$$W_n = \sum_{i=0}^{n} v_i (S_i - \kappa v_i), \qquad \forall n \in \{0, \cdots, N-2\}.$$

The quantity $\bar{x}_n = q_0 - \sum_{i=0}^{n} v_i$ the remaining quantity to sell at step n. Moreover, the VWAP price reads

$$\mathrm{VWAP}_{N-1} = \frac{\sum_{i=0}^{N-1} S_i V_i}{\sum_{i=0}^{N-1} V_i} = \frac{\sum_{i=0}^{N-1} S_i V_i}{Q_N},$$

where $Q_N = \sum_{i=0}^{N-1} V_i$. Thus, the agent optimisation problem is

$$\begin{aligned} &\sup_{(v_s)_{s \geq t} \in \mathcal{A}} \mathbb{E}\left[-\exp(-\gamma(W_{N-2} + v_{N-1}(S_{N-1} - \tilde{\kappa} v_{N-1}) - Q_0 \mathrm{VWAP}_{N-1})\right] \\ &= \sup_{(v_s)_{s \geq t} \in \mathcal{A}} \mathbb{E}\left[-\exp(-\gamma(W_{N-2} + v_{N-1}(S_{N-1} - \tilde{\kappa} v_{N-1}) - \frac{Q_0}{Q_N} \sum_{i=0}^{N-1} S_i V_i)\right]. \end{aligned} \tag{IX.2}$$

Solving the discrete time control problem. We notice that

$$
\begin{aligned}
W_{N-2} &= \sum_{i=0}^{N-2} v_i(S_i - \kappa v_i) \\
&= \sum_{i=0}^{N-2} v_i\big([\sum_{j=0}^{i-1} \beta v_j \Delta + \sigma_j \sqrt{\Delta}\epsilon_{j+1}] + S_0 - \kappa v_i\big) \\
&= \beta\Delta \sum_{i=0}^{N-2} v_i \underbrace{\big(\sum_{j=0}^{i-1} v_j\big)}_{=x_{i-1}} + \sum_{i=0}^{N-2}\sum_{j=0}^{i-1} v_i \sigma_j \sqrt{\Delta}\epsilon_{j+1} + S_0 Q_0 - \sum_{i=0}^{N-2} \kappa v_i^2 \\
&= \beta\Delta \sum_{i=0}^{N-2} v_i x_{i-1} + \sum_{j=0}^{N-3} \epsilon_{j+1}\sigma_j\sqrt{\Delta}\,\underbrace{\big[\sum_{i=j+1}^{N-2} v_i\big]}_{=\bar{x}_j} + S_0 Q_0 - \sum_{i=0}^{N-2} \kappa v_i^2 \\
&= \beta\Delta \sum_{i=0}^{N-2} v_i x_{i-1} + \sum_{j=0}^{N-3} \epsilon_{j+1}\sigma_j\sqrt{\Delta}\bar{x}_j + S_0 Q_0 - \sum_{i=0}^{N-2} \kappa v_i^2.
\end{aligned}
$$

Using the same arguments, we write

$$
\begin{aligned}
v_{N-1}(S_{N-1} - \tilde{\kappa} v_{N-1}) &= \beta\Delta v_{N-1} \sum_{i=0}^{N-2} v_i + v_{N-1} \sum_{i=0}^{N-2} \sigma_i v_i \sqrt{\Delta}\epsilon_{i+1} - \tilde{\kappa} v_{N-1}^2 \\
&= (\beta\Delta x_{N-2} - \tilde{\kappa} v_{N-1}) v_{N-1} + v_{N-1} \sum_{i=0}^{N-2} \sigma_i v_i \sqrt{\Delta}\epsilon_{i+1}.
\end{aligned}
$$

Finally, using the same arguments, the VWAP price reads

$$
\begin{aligned}
Q_0 \mathrm{VWAP}_{N-1} &= \frac{Q_0}{Q_N} \sum_{i=0}^{N-1} S_i V_i \\
&= \frac{Q_0}{Q_N}\big[\beta\Delta \sum_{i=0}^{N-1} V_i x_{i-1} + \sum_{j=0}^{N-2} \epsilon_{j+1}\sigma_j \sqrt{\Delta}\bar{V}_j + S_0 Q_N\big] \\
&= M_0 \beta\Delta \sum_{i=0}^{N-1} V_i x_{i-1} + M_0 \sum_{j=0}^{N-2} \epsilon_{j+1}\sigma_j\sqrt{\Delta}\bar{V}_j + S_0 Q_0,
\end{aligned}
$$

where $\bar{V}_j = \sum_{i=j+1}^{N-1} V_j$ and $M_0 = \frac{q_0}{Q_N}$. Consequently, the optimisation problem reads

$$\begin{aligned}
\sup_{(v_i)_{N-1\geq i}\in\mathcal{A}} & - \exp\big[-\gamma(-\kappa\sum_{i=0}^{N-2} v_i^2 + \beta\Delta\sum_{i=0}^{N-2}(v_i - M_0V_i)x_{i-1} + (\beta\Delta x_{N-2} - \tilde{\kappa}v_{N-1})v_{N-1}\big]\times \\
& \mathbb{E}\left[\exp(-\gamma(\sum_{j=0}^{N-2}\epsilon_{j+1}\sigma_j\sqrt{\Delta}(\bar{x}_j - M_0\bar{V}_j))\right] \\
& = \sup_{(v_i)_{N-1\geq i}\in\mathcal{A}} - \exp\big[-\gamma(-\kappa\sum_{i=0}^{N-2} v_i^2 + \beta\Delta\sum_{i=0}^{N-2}(v_i - M_0V_i)x_{i-1} \\
& + (\beta\Delta x_{N-2} - \tilde{\kappa}v_{N-1})v_{N-1}\big]\times\exp\big[\frac{\gamma^2}{2}\sum_{j=0}^{N-2}\sigma_j^2\Delta(\bar{x}_j - M_0\bar{V}_j)^2\big]
\end{aligned} \tag{IX.3}$$

The first maximization problem is analogous to

$$\begin{aligned}
\sup_{(v_i)_{N-1\geq i}\in\mathcal{A}} & \big[-\kappa\sum_{i=0}^{N-2} v_i^2 + \beta\Delta\sum_{i=0}^{N-2}(v_i - M_0V_i)x_{i-1} + (\beta\Delta x_{N-2} - \tilde{\kappa}v_{N-1})v_{N-1} \\
& - \frac{\gamma}{2}\sum_{j=0}^{N-2}\sigma_j^2\Delta(\bar{x}_j - M_0\bar{V}_j)^2\big]
\end{aligned} \tag{IX.4}$$

Solving the control problem leads to the following equations

$$\begin{cases} a_ix_i + b_ix_{i+1} + c_ix_{i-1} - m_i = 0 & \forall i \in \{0, \cdots, N-1\} \\ x_{N-1} = Q_0, \end{cases}$$

where $a_i = -4\kappa - 2\beta\Delta + (\gamma\sigma_i)^2\Delta$, for $i < N-2$, $a_{N-2} = -2(\kappa+\tilde{\kappa}) - 2\beta\Delta + (\gamma\sigma_{N-2})^2\Delta$ and $a_{N-1} = -2\tilde{\kappa}$, $b_i = 2\kappa + \beta\Delta$ for $i < N-2$, $b_{N-2} = 2\tilde{\kappa} + \beta\Delta$ and $b_{N-1} = 0$, $c_0 = 0$, $c_i = 2\kappa + \beta\Delta$ for $i < N-1$, and $c_{N-1} = 2\tilde{\kappa} + \beta\Delta$, $m_i = \beta\Delta M_0V_{i+1} + \gamma\sigma_i^2\Delta M_0Q_i$, for all $i < N-2$, $m_{N-2} = \gamma\sigma_i^2\Delta M_0Q_i$ and $m_{N-1} = 0$. Thus, we have to solve the linear system

$$AX = M,$$

where A is tridiagonal matrix where a_i are the diagonal coefficients and sub/super diagonal coefficients are filled with the constant value b. The symmetric matrix A is always diagonalizable.

Remark 2.1. *We can play with values of the parameter $\tilde{k}$ to ensure that $x_N = 0$.*

2.2 Time continuous framework

Let $T < \infty$ the final trade horizon. Suppose that, at initial time 0, an institutional broker decides to sell a quantity $Q_0 > 0$ of a tradable asset using a VWAP based strategy. Hence, he has to control his trading speed ν_t. The dynamics of the asset reference price S_t has the following components. First, an exogenous movement generated by the standard Brownian motion $(W_t)_{t\in[0,T]}$ (with its natural probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ and the adapted filtration $(\mathcal{F}_t)_{t\in[0,T]}$). Second, a permanent linear price impact generated by the agent's trading activity, see [7]. Hence, S_t satisfy

$$dS_t = \beta v_t dt + \sigma dW_t,$$

where $\beta > 0$ is a positive homogenization parameter. The agent's state is characterized by two variables. That is, his inventory, Q_t, and his wealth, X_t. Since he has q_0 shares to liquidate, we have $Q_0 = q_0$. The trader's inventory Q_t evolves according to

$$dQ_t = v_t dt.$$

We notice that ν_t is negative for a seller. The wealth X_t is affected by a temporary linear market impact $\kappa\nu_t$

$$dX = -\nu_t\big[S_t + \kappa\nu_t)\big]dt.$$

Finally, we define the VWAP at time T by

$$\text{VWAP}_T = \frac{\int_0^T S_t V_t dt}{\int_0^T V_t} = \frac{\int_0^T S_t V_t dt}{Q_T}. \tag{IX.5}$$

The agent's value function is

$$\begin{aligned}\bar{V}(t,s,q,x) &= \sup_{(v_s)_{s\geq t}\in\mathcal{A}} \mathbb{E}\left[-\exp(-\gamma(X_T + Q_T(S_T - \tilde{\kappa}Q_T) - q_0\text{VWAP}_T)\right] \\ &= \sup_{(v_s)_{s\geq t}\in\mathcal{A}} \mathbb{E}\bigg[-\exp(-\gamma(X_T + Q_T(S_T - \tilde{\kappa}Q_T) \\ &\qquad -\frac{q_0}{Q_T}\int_0^T S_t V_t dt)\bigg].\end{aligned} \tag{IX.6}$$

The trader's objective function (IX.6) is the expected CARA[3] utility function on the terminal cash. Moreover, we consider the VWAP as the liquidation penalty (see [25] for discussions on the liquidation penalty function).

Solving the control problem. The Hamilton-Jacobi-Bellman equation associated to IX.6 is

$$\begin{cases} \partial_t\bar{V} + \frac{\sigma^2}{2}\partial_{ss}\bar{V} + \gamma M_0 s V_t \bar{V} + \sup_\nu\left[\nu(\partial_q\bar{V} + \beta\partial_s\bar{V}) - \nu(s+\kappa\nu)\partial_X\bar{V}\right] = 0 \\ \bar{V}(T,s,q,x) = -\exp(-\gamma(x + q(s-\tilde{\kappa}q)), \end{cases}$$

where $M_0 = \frac{q_0}{Q_T}$. To solve the equation, we use the following ersatz :

$$\bar{V}t,s,q,x) = -\exp\big[-\gamma(x + q(s-\tilde{\kappa}q) - M_0 V_t s(T-t))\big]v(t,q)$$

Hence, $v(t,q)$ satisfy the HJB equation

$$\begin{cases} \partial_t v + \frac{\left(\sigma\gamma\left[q - M_0 V(T-t)\right]\right)^2}{2} v + \sup_\nu\left[\nu(\partial_q v - \gamma\beta(q - M_0 s V(T-t))v) + \nu^2\kappa\gamma v\right] = 0 \\ v(T,q) = 1. \end{cases} \tag{IX.7}$$

[3]Constant Absolute Risk Aversion

The associated optimal control is

$$\nu^* = -\frac{\partial_q v - \gamma\beta(q - M_0 V(T-t))v}{2\kappa\gamma v}.$$

Thus, by inserting ν^* expression in IX.7, we get

$$\begin{cases} \partial_t v + \frac{\left(\sigma\gamma\left[q - M_0 V(T-t)\right]\right)^2}{2} v - \frac{\left(\partial_q v - \gamma\beta\left[q - M_0 V(T-t)\right]v\right)^2}{4\kappa\gamma v} = 0 \\ v(T,q) = 1. \end{cases} \tag{IX.8}$$

When v can be written in the following form

$$v(t,q) = e^{h_0(t) + h_1(t)q + \frac{h_2(t)}{2}q^2},$$

it is easy to check, using equation IX.7, that h_0, h_1 and h_2 satisfy the system of equations

$$\begin{cases} h_2' + \frac{\beta}{\kappa}h_2 - \frac{1}{2\kappa\gamma}h_2^2 = -\left(\sigma\gamma\right)^2 + \frac{\gamma\beta^2}{2\kappa} \\ h_1' + \frac{h_2 - \gamma\beta}{2\kappa\gamma}h_1 = M_0 V(T-t)\left[\frac{h_2 - \gamma\beta}{2\kappa\gamma} - \sigma\gamma\right] \\ h_0' = -\frac{\left(\sigma\gamma M_0 V(T-t)\right)^2}{2} - \frac{[h_1 - M_0 V(T-t)]^2}{4\kappa\gamma} \end{cases}$$

with the terminal condition

$$h_0(T) = h_1(T) = h_2(T) = 0.$$

the solution h_2 of the first equation has the following form

$$h_2(t) = \frac{1}{a - (a - \frac{1}{C})e^{r(T-t)}} - C,$$

where $a = \frac{1}{2\gamma\beta}$, $r = \frac{\beta}{\kappa}$ and $C = \beta\gamma + \sigma\gamma\sqrt{2\kappa\gamma}$. Given the expression of h_2, $h_1 = 0$ is solution of the second equation. Using the expression of h_1, the solution h_0 of the third equation is

$$h_0(t) = -\frac{(M_0 V)^2}{6}[\frac{1}{2\kappa\gamma} + (\sigma\gamma)^2](T-t)^3.$$

Finally, using equation 2.2, we have

$$\bar{V}(t,s,q,x) = -\exp\left[-\gamma\left(x + q(s - \tilde{\kappa}q) - M_0 V_t s(T-t)\right) + h_0(t) + \frac{h_2(t)}{2}q^2\right]$$

Using verification arguments we have the existence and uniqueness of the solution.

Chapter X

Market simulation

The understanding of the market activity in its finest time scale is a challenging task. The reason is that different classes of market participants (agents) take part in the dynamics of an asset price. Besides, each market participant may have a different utility and risk profile, among other particularities. Here, we regard three agent classes acting dynamically in a limit order book of a financial asset. Namely, we consider market makers (MM), high-frequency trading (HFT) firms, and institutional brokers (IB). The MM provides liquidity to other market participants using limit orders. But they may also send market orders to get immediate execution. Additionally, the HFT goal is to identify profit opportunities to catch or at the opposite critical situations to avoid. To do this, they use limit/market orders and exploit the correlation with another asset. Finally, the IB has a prescheduled task to sell or buy many shares of the considered asset. The IB use only market orders to acquire or liquidate the needed shares. In our setting, the main sources of risk are: the uncertainty of price variation, the impact of limit/market orders on the order book state (i.e available liquidity and prices at the best limits) and the inventory risk. Since the order book dynamics rises from the interaction between different types of agents acting optimally, we compute the optimal strategy of each agent and then use these optimal strategies to simulate the market activity. The agent's strategies rely on the modeling framework of stochastic optimal control theory. Furthermore, our setting is consistent with the recent literature on market microstructure, especially in algorithmic and HFT.

Here, we present the simulation methodology of the behavior of market participants acting together in the limit order book of the traded asset.

General description:

(i) **Market makers.** We consider m market makers that play in a random order at each "time block". Market makers strategy optimal strategy adapts to the changes in the order book state.

(ii) **HFT firms.** We choose n high frequency traders and let them play in a random order at each block of time. High frequency traders have priority access to the market, they react to the current state of the order book and do it before the market makers.

(iii) **Institutional brokers.** We consider r robot traders with a predefined strategy : they have to liquidate or to acquire a certain quantity of the stock for economical reasons. Robot traders take their decisions after the market makers.

Result: Simulation of a limit order book with placement of orders from market makers, high frequency traders, and institutional brokers.

$n < m$;
$MarketMakers = \{MM^1, MM^2, \ldots, MM^m\}$;
$HFTFirms = \{HFT^1, HFT^2, \ldots, HFT^n\}$;
$InstitutionalBroker = \{IB^1\}$
$AGENTS = \{MarketMakers, HFTFirms\}$
$t = 1$;
while $t < T$ **do**
  $MarketMakers = RandomlyOrder(MarketMakers)$;
  $HFTraders = RandomlyOrder(HFTraders)$;
  $beginState = X_t$;
  $currentState = beginState$;
  for $i = 1$ *to* n **do**
    $\Delta\alpha_t^{i,*} = GetOptimalControlHFT(currentState, HFT^i, \delta)$;
    $currentState = UpdateLOB(currentState, \Delta\alpha_t^{i,*}, \delta)$;
    $\Delta\zeta_t^{i,*} = GetOptimalControlMM(beginState, MM^i, \delta)$;
    $currentState = UpdateLOB(currentState, \Delta\zeta_t^{i,*}, \delta)$;
  end
  for $i = n + 1$ *to* m **do**
    $\Delta\alpha_t^{i,*} = GetOptimalControlHFT(currentState, HFT^i, \delta)$;
    $currentState = UpdateLOB(currentState, \Delta\alpha_t^{i,*}, \delta)$;
  end
  $\Delta Z_t = SimulateOrderArrival(currentState, \delta)$;
  $currentState = UpdateLOB(currentState, \Delta Z_t, \delta)$;
  $t + 1$
end

Algorithm 1: Market simulation with market makers, high frequency traders and exogenous traders.

We have,

(i) $SimulateOrderArrival$: This function represents an order from other market participants. It is simulated depending on the current state of the limit order book and the tick size. Here, ΔZ_t is the output of an order arrival at time t .

(ii) $GetOptimalControlMM$: This function represents the market maker optimal control. The numerical solution of this problem is detailed in section **??**. The output at time t is $\Delta\zeta_t^*$.

(iii) $UpdateLOB$: This function represents the placement of a given oder. It updates the state of the LOB, depending if the order comes from either a market maker or an exogenous trader.

CHAPTER XI

Appendix

1 Ergodicity of the Order book dynamics

We provide a Ergodicity results on the order book dynamics $U := (Q^{\mathfrak{b}}, Q^{\mathfrak{a}}, P^{\mathfrak{b}}, P^{\mathfrak{a}})$ which ensures the order book will convergence to some invariant probability. This serves as a basis for validation of the arguments that the stylized fact on market data can be explained by this invariant distribution in the limit case.

Assumption 2. *(Negative individual drift) There exist three positive constants C_{boudnd}, $z_0 > 1$ and $\delta > 0$ such that for any $u = (q^b, q^a, p^b, p^a)$ and $i \in \{\mathfrak{a}, \mathfrak{b}\}$ with $q^i \geq C_{bound}$,*

$$\sum_{k\geq 0}(z_0^n - 1)(\lambda_L^i(k) - \lambda_A^i(k)\frac{1}{z_0^n}) \leq -\delta$$

Assumption 3. *(Bounded on incoming flow) There exist a positive constants H, such that for any $u = (q^b, q^a, p^b, p^a)$ and $i \in \{\mathfrak{a}, \mathfrak{b}\}$,*

$$\sum_{k\geq 0} \lambda_L^i(k) + \lambda_A^i(k) \leq H$$

Assumption 4. *(Regeneration bound) There exist three positive constants C^{Regen}, L and $z_1 > 1$, such that for any $u = (q^b, q^a, p^b, p^a)$ and $i \in \{\mathfrak{a}, \mathfrak{b}\}$*

$$\mathbb{E}\left[\sum_{i=1}^{2} z_1^{|R^i(u;)-C^{Regen}|}\right] < L$$

Theorem 1.1. *(Ergodicity) Under the following assumptions, the process $(U_t) = (Q_t^{\mathfrak{b}}, Q_t^{\mathfrak{a}}, P_t^{\mathfrak{b}}, P_t^{\mathfrak{a}})$ is ergodic. i.e. converges towards a unique invariant distribution.*

The proof principle Let $\mathcal{Q}$ the infinitesimal generator of the process (U_t). To prove that U_t is ergodic, we design a Lyapunov function $V : \mathbb{N}^2 \times \mathbb{R}^2 \to (0, \infty)$, on which the negative drift condition is satisfied for some $c > 0$ and $d > 0$:

$$\mathcal{Q}V(u') \leq -cV(u') + d.$$

Then, using Theorem 6.1 in Meyn and Tweedie (1993), the Markov process U is non-explosive and V-uniformly ergodic. Furthermore, by Theorem 4.2 in Meyn and Tweedie (1993) it is Harris positive recurrent. This methodology is also used in [4, 31].

proof Let $z > 1$, $C^{'}_{bound} = \max(C_{bound}, C^{Regen})$. For any $u' = (q^{\mathfrak{b}}, q^{\mathfrak{a}}, p^{\mathfrak{b}}, p^{\mathfrak{a}})$, we have

$$V(u') = \sum_{i \in \{\mathfrak{a},\mathfrak{b}\}} z^{|q^i - C^{'}_{bound}|_+} + \sum_{i \in \{\mathfrak{a},\mathfrak{b}\}} z^{|p^i - C^{'}_{bound}|_+}.$$

First, we let $\mathcal{Q}$ the infinitesimal generator of the process (U_t):

$$\begin{aligned}
\mathcal{Q}V(u^{'}) &= \sum_{u' \neq w'} \mathcal{Q}_{u^{'},w^{'}} \left[V(w^{'}) - V(u^{'})\right] \\
&= \sum_{i \in \{\mathfrak{a},\mathfrak{b}\}} \sum_{1 \leq n < q^i} \overbrace{\left[\lambda^i_L(n)(z^{|q^i + n - C^{'}_{bound}|_+} - z^{|q^i - C^{'}_{bound}|_+})\right.}^{(1)\text{ No price move}} \\
&\qquad \left. + \overbrace{\lambda^i_A(n)(z^{|q^i - n - C^{'}_{bound}|_+} - z^{|q^i - C^{'}_{bound}|_+})}^{(1)\text{ No price move}}\right] \\
&+ \sum_{i \in \{\mathfrak{a},\mathfrak{b}\}} \sum_{n \geq q^i} \overbrace{\left[\lambda^i_L(n)(z^{|q^i + n - C^{'}_{bound}|_+} - z^{|q^i - C^{'}_{bound}|_+})\right]}^{(2)\text{ Large limit orders insertion}} \\
&+ \sum_{i \in \{\mathfrak{a},\mathfrak{b}\}} \sum_{n \geq q^i} \overbrace{\left[\lambda^i_A(n)(\mathbb{E}\left[V\left(R^i(u';)\right)\right] - V(u'))\right]}^{(3)\text{ order book regeneration}}
\end{aligned}$$

Next, we study the contribution of each part of the above sum.

- **Case 1:** When there is no price move, we have :

$$\begin{aligned}
(1) &= \lambda^i_L(n)(z^{|q^i + n - C^{'}_{bound}|_+} - z^{|q^i - C^{'}_{bound}|_+}) \\
&\quad + \lambda^i_A(n)(z^{|q^i - n - C^{'}_{bound}|_+} - z^{|q^i - C^{'}_{bound}|_+}) \\
&= \lambda^i_L(n)\mathbf{1}_{q^i \geq C^{'}_{bound}} z^{|q^i - C^{'}_{bound}|_+}(z^n - 1) \\
&\quad + \lambda^i_A(n)\mathbf{1}_{q^i \geq C^{'}_{bound} + n} z^{|q^i - C^{'}_{bound}|_+}(\frac{1}{z^n} - 1) \\
&= (z^n - 1) z^{|q^i - C^{'}_{bound}|_+} \left[\lambda^i_L(n)\mathbf{1}_{q^i \geq C^{'}_{bound}} - \frac{\lambda^i_A(n)\mathbf{1}_{q^i \geq C^{'}_{bound} + n}}{z^n}\right].
\end{aligned}$$

  Using Assumptions 2 and 3, for z sufficiently close to 1, we have

$$\lambda^i_L(n)\mathbf{1}_{q^i \geq C^{'}_{bound}} - \frac{\lambda^i_A(n)\mathbf{1}_{q^i \geq C^{'}_{bound} + n}}{z^n} \leq z^{-n}\left(-\delta + H(z^n - 1)\right) \leq -\delta^{''} < 0.$$

  Consequently, we obtain

$$(1) \leq -\delta^{''}(z^n - 1) z^{|q^i - C^{'}_{bound}|_+}.$$

- For Part (2) (i.e large limit orders of size $n > q^i$), we keep the same expression.
- **Case 3:** When the order book is regenerated, we use Assumption 4 to obtain

$$\begin{aligned}(3) &= \lambda^i_A(n)\big[\mathbb{E}\left[V\big(R^i(u';)\big)\right] - V(u')\big]\\ &\leq \lambda^i_A(n)L - \lambda^i_A(n)V(u').\end{aligned}$$

Finally, by combining above inequalities we have

$$\begin{aligned}\mathcal{Q}V(u^{'}) &\leq -\delta^{''}(z^{q^i}-1)V(u')\big[\sum_{i\in\{\mathfrak{a},\mathfrak{b}\}}\sum_{1\leq n<q^i}1\big] + \big[L^J - V(u')\big[\sum_{i\in\{\mathfrak{a},\mathfrak{b}\}}\sum_{n\geq q^i}\lambda^i_L(n)\big]\big]\\ &+ \big[HL - V(u')\big[\sum_{i\in\{\mathfrak{a},\mathfrak{b}\}}\sum_{1\leq n\geq q^i}\lambda^i_A(n)\big]\big]\\ &\leq -\delta^{(3)}V(u') + (HL + L^J),\end{aligned}$$

where $\delta^{(3)} = \delta^{''}(z^{q^i}-1)2q^i + \big[\sum_{1\leq n\geq q^i}\lambda^i_A(n) + \lambda^i_L(n)\big]$.

2 Numerical scheme error estimate

Notations : We define as well the continous $\bar{Y}^{h,0,y,\zeta}_t$ process associated to $Y^{h,0,y,\zeta}_t$ such that:

$$\begin{cases}\bar{Y}^{h,0,y,\zeta}_{kh} = \tilde{Y}^{h,0,y,\zeta}_k\\ \bar{Y}^{h,0,y,\zeta}_t = \tilde{Y}^{h,0,y,\zeta}_k + \frac{t-kh}{h}(\tilde{Y}^{h,0,y,\zeta}_{k+1} - \tilde{Y}^{h,0,y,\zeta}_k) \quad \forall t\in(kh,(k+1)h).\end{cases}$$

We note $\bar{V}^h(t,y)$ the utility function of the control problem associated to the process $\bar{Y}^{h,0,y,\zeta}_t$.

Step 1: First, we show that for every initial state $y = (i,n,b,s,q)$ and $t \leq T$, the following estimate holds.

$$|\bar{V}^h(t,y) - V(t,y)| \leq R'(T-t)h, \tag{XI.1}$$

where $R' = ||Q||C_{Lip}(1+y)\lambda^{y,*}$ and $\lambda^{y,*} = \sum_{i\in\{\mathfrak{a},\mathfrak{b}\};q}\lambda^{y,i}_L(q) + \lambda^{y,i}_A(q)$.

Proof of inequality XI.1: Let us fix h and show the result by recurrence on n for every $T\in[0,nh]$. To simplify, we denote V_T the utility function with terminal time T.

- *Initialization:* Case $n=0$, in this case we have $\bar{V}^h_T(0,y) = V_T(0,y) = w(y)$ for every initial state y.
- *Iteration:* let us assume the result true for every $T^{'}\in[0,nh]$. Let $T\in[0,(n+1)h]$.
  - *If* $T\in[0,nh]$*:* the result is true using the recurrence assumption.
  - *If* $T\in(nh,(n+1)h]$*:* let $t\in[0,T]$. When $t\in[h,T]$, using $V_T(t,y) = V_{T-t}(0,y)$, $\bar{V}^h_T(0,y) = \bar{V}^h_{T-t}(0,y)$ and the recurrence hypothesis, we have:

$$|\bar{V}^h(t,y) - V_T(t,y)| = |\bar{V}^h_{T-t}(0,y) - V_{T-t}(0,y)| \leq R'(T-t)h,$$

which proves the result for $t \in [h,T]$.
When $t \in [0,h)$, we are going to build the Markov chain $\tilde{Y}_n^{h,0,y,\zeta}$ using the order book dynamic $Y_t^{0,y,\zeta}$.

Construction of the Markov chain $\tilde{Y}_n^{h,0,y,\zeta}$: The process $Y_t^{0,y,\zeta}$ is defined on the probability $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Let $(\Omega^{'}, \mathcal{F}^{'}, \mathbb{P}^{'})$ be a new probability space. Let us consider $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t), \bar{P})$ the product space, T_i the jump times of the process U, N_t the number of jumps up to time t. Consider the set

$$A_n^h = \big\{\omega \in \Omega,\ s.t\, T_{n+1}(\omega) - T_n(\omega) \geq h\big\}.$$

Finally, we note $N_t^T = N_T - N_t$ the number of jumps between t and T.
Note that

$$\mathbb{P}[A_n^h] = \mathbb{P}[T_{n+1} - T_n \geq h] = 1 - \big(\mathbb{P}[N_{T_n}^h = 1] + \mathbb{P}[N_{T_n}^h \geq 2]\big),$$

and

$$\mathbb{P}[N_{T_n}^h = 1] = \sum_q \big(p_1(q) + p_2(q) + p_3(q) + p_4(q)\big).$$

Thus, using that $p_5 = 1 - \mathbb{P}[N_{T_n}^h = 1]$, we have $P[A_n^h] \leq p_5$ and $P[A_n^{h,c}] \geq \mathbb{P}[N_{T_n}^h = 1] = \sum_{1 \leq i \leq 5; q} p_i(q) > 0$.
Next, we are going to use the space $\Omega^{'}$. There exists, $\Omega_i^n(q) \in \mathcal{F}^{'}$ with $1 \leq i \leq 5$, such that $\sum_{1 \leq i \leq 4; q} \mathbb{P}\big[\Omega_i^n(q)\big] + \mathbb{P}(\Omega_5^n) = 1$ and $\mathbb{P}(\Omega_i^n(q)) = p_i(q)/P[A_n^{h,c}]$ when $i \leq 4$ and $\mathbb{P}(\Omega_5^n) = (p_5 - P[A_n])/P[A_n^c]$. Finally, we define $\tilde{Y'}_n^{h,0,y,\zeta}$ recursively, by

$$\begin{cases} \tilde{Y'}_0^{h,0,y,\zeta} = y \\ \tilde{Y'}_{n+1}^{h,0,y,\zeta} = \mathbf{1}_{A_n^h} \tilde{Y'}_n^{h,0,y,\zeta} + \mathbf{1}_{A_n^{h,c}} \sum_{1 \leq i \leq 5; q} \mathbf{1}_{\Omega_i^n}(q) \tilde{Y'}_n^{h,0,y,\zeta}(i,q), \end{cases}$$

where $\tilde{Y'}_n^{h,0,y,\zeta}(i,q)$ is the new state of the order book whe the event associated to the probability $p_i(q)$ happens. For instance, $\tilde{Y'}_n^{h,0,y,\zeta}(1,q)$ is the new state when one quantity q is added to the bid side. With this construction, we have two properties: first, $\tilde{Y'}_n^{h,0,y,\zeta}$ have the same law of the discrete Markov chain $\tilde{Y}_n^{h,0,y,\zeta}$; second, $\tilde{Y'}_n^{h,0,y,\zeta} \mathbf{1}_{A_n^h} = \tilde{Y'}_n^{h,0,y,\zeta}$.

Continuation of the proof: We note T_c the first jump of $Y_t^{h,0,y,\zeta}$ and $\tau = (T_c - t) \wedge h$. Using the dynamic programming principle, we have

$$\begin{aligned} \big|\bar{V}^h(t,y) - V(t,y)\big| &= \big|\bar{V}_{T-t}^h(0,y) - V_{T-t}(0,y)\big| \\ &\leq \big|\sup_\zeta \mathbb{E}\left[\bar{V}_{T-t}^h(\tau, \bar{Y}_\tau^{h,\zeta}) - V_{T-t}(\tau, Y_\tau^\zeta)\right]\big| \\ &= \Big|\sup_\zeta \mathbb{E}\Big[\int_0^{T_c - t} Q(\bar{V}_{T-t}^h(s,y) - V_{T-t}(s,y)\mathrm{d}s \mathbf{1}_{T_c - t < h} \\ &\quad + \big[\bar{V}_{T-t}(h,y) - V_{T-t}(h,y)\big]\mathbf{1}_{T_c - t \geq h}\Big]\Big| \end{aligned}$$

* First, we have:

$$
\begin{aligned}
&\mathbb{E}\Big|\int_0^{T_c-t} Q[\bar{V}^h_{T-t}(s,y) - V_{T-t}(s,y)]\mathrm{d}s\mathbf{1}_{T_c-t<\Delta}\Big| \\
&\qquad\le ||Q||\mathbb{E}\left[\int_0^{T_c-t} |\bar{V}^h_{T-t}(s,y) - V_{T-t}(s,y)|\mathbf{1}_{T_c-t<h}\right] \\
&\qquad\le ||Q||\mathbb{E}\left[\int_0^{T_c-t} C(1+y)\sqrt{T-s}\mathbf{1}_{T_c-t<h}\right] \\
&\qquad\le ||Q||C(T-t)h^2\lambda^{y,*}.
\end{aligned}
$$

With $C = C_{Lip}(1+y)$, and $\lambda^{y,*} = \sum_{i\in\{\mathfrak{a},\mathfrak{b}\};q}\lambda^{y,i}_L(q) + \lambda^{y,i}_A(q)$. In the second inequality, we use that

$$
\begin{aligned}
|\bar{V}^h_{T-t}(s,y) - V_{T-t}(s,y)| &\le |\bar{V}^h_{T-t}(s,y) - \bar{V}_{T-t}(T,y)| \\
&\quad + |V_{T-t}(s,y) - V_{T-t}(T,y)|
\end{aligned}
$$

and

$$V_{T-t}(T,y) = \bar{V}^h_{T-t}(T,y) = U(y).$$

Moreover, we use that V and $\bar{V}^h$ are $\frac{1}{2}$-Holder in time because the final constraint is Lipschitz.

* Finally, using the recurrence assumption, we have

$$\mathbb{E}\left[\bar{V}_{T-t}(h,U) - V_{T-t}(h,U))\mathbf{1}_{T_c-t\ge h}\right] \le R'(T-t-h)h.$$

Consequently, we have :

$$
\begin{aligned}
|\bar{V}^h(t,y) - V(t,y)| &\le R(T-t-h)h + ||Q||C(T-t)h^2\lambda^{y,*} \\
&\le R(T-t)h.
\end{aligned}
$$

Which proves the result.

Step 2 : Let us show for every initial state and $t\le T$, we have

$$|\bar{V}^h(t,y) - V^h(t,y)| \le R^{(2)}h, \tag{XI.2}$$

with $R^{(2)} = U_{Lip}(2L^j)$.

Proof of inequality XI.2 First, we note that

$$|\bar{V}^h(t,y) - V^h(t,y)| \le \sup_\zeta \mathbb{E}\left[U(\bar{Y}^{h,\zeta}_T) - U(Y^{h,\zeta}_T)\right].$$

We then use that U is Lipschitz to obtain

$$\begin{aligned}
|\bar{V}^h(t,U) - V^h(t,U)| &\leq U_{Lip} \sup_{\zeta} \mathbb{E}\left[|\bar{Y}_T^{h,\zeta} - Y_T^{h,\zeta}|\right] \\
&\leq U_{Lip} \sup_{\zeta} \mathbb{E}\left[|\tilde{Y}_{n_T+1}^{h,\mu} - \tilde{Y}_{n_T}^{h,\mu}| \mathbf{1}_{\tilde{Y}_{n_T}^{h,\mu} \neq \tilde{Y}_{n_T+1}^{h,\mu}}\right] \\
&\leq U_{Lip}(2L^j)h.
\end{aligned}$$

Where L^j is defined in Assumption **??**.

Finally, by combining (XI.1) and (XI.2), we prove (**??**) with $R = R^{(2)} \times R'$.

3 Model parameters estimation

The estimation method is inspired from [29]. We first show how to estimate the intensities and then the regeneration process.

3.1 Intensities estimation

Here, we focus on the case of the first bid limit since other cases are estimated in the same way. We assume that intensities depends only on the quantities available at the best limits and the spread value. Hence, we replace the initial state y by $\tilde{y} = (q^b, q^a, s)$. Let $t \in \{A, L\}$ denote the order type, and $k \in \{1, \cdots, M\}$ the order size. The methodology estimation consists in the following. First, we estimate limit and aggressive orders intensities without considering the order size $\lambda_t^{\tilde{y},\mathfrak{b}}$. Next, we estimate the order size probability, $p_t^{\tilde{y},\mathfrak{b}}(k)$. By combining both estimations, we have $\lambda_t^{\tilde{y},\mathfrak{b}}(k) = \lambda_t^{\tilde{y},\mathfrak{b}} p_t^{\tilde{y},\mathfrak{b}}(k)$.

3.1.1 Non parametric estimation

Notation : We note an event "$\omega_\mathfrak{b}$" any modification of the bid size. For every couple (Q^b, Q^a, S), we store the waiting time $\Delta_t(\omega_\mathfrak{b})$ (in number of seconds) between the event "$\omega_\mathfrak{b}$" and the preceeding event, the type of event $\mathcal{T}(\omega_\mathfrak{b})$, the bid size, ask size and the spread value $(q^b(\omega_\mathfrak{b}), q^a(\omega_\mathfrak{b}), s(\omega_\mathfrak{b}))$ before the event. We consider two sets for the event type $\mathcal{T}(\omega_\mathfrak{b})$:

- $\mathcal{T}(\omega_\mathfrak{b}) \in \xi^L$ for limit orders insertion at Q^b.
- $\mathcal{T}(\omega_\mathfrak{b}) \in \xi^A$ for aggressive orders at Q^b.

The queue size is approximated by the least greater integer and divided by the stock average event size.

Intensities estimation: Once $\big(\Delta_t(\omega_\mathfrak{b}), \mathcal{T}(\omega_\mathfrak{b}), q^{bid}(\omega_\mathfrak{b}), q^{ask}(\omega_\mathfrak{b}), s(\omega_\mathfrak{b})\big)$ computed, we estimate $\hat{\lambda}_t^{\tilde{y},\mathfrak{b}}$ with $\tilde{y} = (m, n, s)$:

$$\begin{aligned}
\hat{\Lambda}^{\tilde{y}} &= \big(\mathrm{mean}(\Delta_t(\omega_\mathfrak{b}) | Q^b(\omega_\mathfrak{b}) = m, Q^a(\omega_\mathfrak{b}) = n, S(\omega_\mathfrak{b}) = s)\big)^{-1} \\
\hat{\lambda}_t^{\tilde{y},\mathfrak{b}} &= \hat{\Lambda}^{\tilde{y}} \frac{\#\big\{\mathcal{T}(\omega_\mathfrak{b}) \in \xi^t | q^b(\omega_\mathfrak{b}) = m, q^a(\omega_\mathfrak{b}) = n, s(\omega_\mathfrak{b}) = s\big\}}{\#\big\{q^b(\omega_\mathfrak{b}) = m, q^a(\omega_\mathfrak{b}) = n, s(\omega_\mathfrak{b}) = s\big\}}
\end{aligned}$$

Where mean denotes the empirical mean and $\#A$ the cardinality of the set A. Given the symetry relation, the bid and ask side are aggregated to get more data.

Intensities confidence intervals computation: Since arrival and cancellation times are independent, using the central limit theorem, we have, with asymptotic probability 95% $\big(\hat{p}_t^{\tilde{y},\mathfrak{b}} = \frac{\#\left\{\mathcal{T}(\omega_\mathfrak{b})\in\xi^t | q^b(\omega_\mathfrak{b})=m, q^a(\omega_\mathfrak{b})=n, s(\omega_\mathfrak{b})=s\right\}}{\#\left\{q^b(\omega_\mathfrak{b})=m, q^a(\omega_\mathfrak{b})=n, s(\omega_\mathfrak{b})=s\right\}}\big)$:

$$\begin{array}{lcll} \Lambda^{\tilde{y}} & \in & \Big[\hat{\Lambda}^{\tilde{y}} \pm \frac{1.96\hat{\Lambda}^{\tilde{y}}}{\sqrt{\#\left\{q^b(\omega_\mathfrak{b})=m, q^a(\omega_\mathfrak{b})=n, s(\omega_\mathfrak{b})=s\right\}}}\Big] & = & [\hat{\Lambda}^{\tilde{y}}_{\min}, \hat{\Lambda}^{\tilde{y}}_{\max}] \\ \frac{\hat{\lambda}_t^{\tilde{y},\mathfrak{b}}}{\Lambda^{\tilde{y}}} & \in & \Big[\hat{p}_t^{\tilde{y},\mathfrak{b}} \pm \frac{1.96\sqrt{\hat{p}_t^{\tilde{y},\mathfrak{b}}(1-\hat{p}_t^{\tilde{y},\mathfrak{b}})}}{\sqrt{\#\left\{q^{bid}(\omega_i)=n, q^{ask}(\omega_i)=m\right\}}}\Big] & = & [\hat{p}^{\tilde{y},\mathfrak{b}}_{t,\min}, \hat{p}^{\tilde{y},\mathfrak{b}}_{t,\max}]. \end{array}$$

Consequently, with at least 90% :

$$\hat{\lambda}_t^{\tilde{y},\mathfrak{b}} \quad \in \quad \big[\hat{\Lambda}^{\tilde{y}}_{\min}\hat{p}^{\tilde{y},\mathfrak{b}}_{t,\min}, \hat{\Lambda}^{\tilde{y}}_{\max}\hat{p}^{\tilde{y},\mathfrak{b}}_{t,\max}\big] \ .$$

Order size estimation: Let $V_t^{\tilde{y},\mathfrak{b}}$ the order size associated to the event ξ^t. The empirical distribution of the variables $V_t^{\tilde{y},\mathfrak{b}}$ is given by

$$\begin{cases} \mathbb{P}\big[V_t^{\tilde{y},\mathfrak{b}} \in (x,y]\big] \approx \hat{F}_N(y) - \hat{F}_N(x) \\ \hat{F}_N(x) = \dfrac{1}{N}\sum_{i=1}^N \mathbf{1}_{V_t^{\tilde{y},\mathfrak{b},i} \le x}. \end{cases}$$

Where $V_t^{\tilde{y},\mathfrak{b},i}$ are N i.i.d random variables with the same ditribution as $V_t^{\tilde{y},\mathfrak{b}}$.

3.1.2 Parametric estimation

We denote $\mathrm{Imb}_t = \dfrac{Q_t^\mathfrak{b} - Q_t^\mathfrak{a}}{Q_t^\mathfrak{b} + Q_t^\mathfrak{a}}$ the imbalance process. For the parametric estimation, we assume that limit and aggressive orders intensities are linear functions of the imbalance (see [12, 35]). Since orders arrival times are independent and follow an exponential distribution, we can use the Generalized Linear Models framework to estimate model's parameter. The maximum likelihood estimator used for the calibration is consistent and asymptotically normal (see [20]).

3.2 Regeneration process

We focus in the case of the first bid limit when the spread equals 2 tick since other cases are estimated in the same way. We note $Q_\mathfrak{b}^{\mathfrak{b};\tilde{y},Disc}$ and $Q_\mathfrak{b}^{\mathfrak{a};\tilde{y},Disc}$ respectively the discovered quantities at the bid and ask side when the first bid is totally consumed. An event, $\omega_\mathfrak{b}^{move}$, is any decrease in the bid price. For every $\omega_\mathfrak{b}^{move}$, we store the the bid size, ask size and spread before the event $(q^b(\omega_\mathfrak{b}^{move}), q^a(\omega_\mathfrak{b}^{move}), s(\omega_\mathfrak{b}^{move}))$, and the discovered quantities $Q_\mathfrak{b}^{\mathfrak{b};\tilde{y},Disc}$ and $Q_\mathfrak{b}^{\mathfrak{a};\tilde{y},Disc}$.
Once the quantities $\big(q^b(\omega_\mathfrak{b}^{move}), q^a(\omega_\mathfrak{b}^{move}), s(\omega_\mathfrak{b}^{move})\big)$ collected, we estimate the empirical joint

distribution of $Q_{\mathfrak{b}}^{\tilde{y},Disc} = \left(Q_{\mathfrak{b}}^{\mathfrak{b};\tilde{y},Disc}, Q_{\mathfrak{b}}^{\mathfrak{a};\tilde{y},Disc}\right)$:

$$\mathbb{P}\left[Q_{\mathfrak{b}}^{\tilde{y},Disc} \in \mathrm{I}_1 \times \mathrm{I}_2\right] \approx \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{Q_{\mathfrak{b}}^{\tilde{y},Disc,i} \in \mathrm{I}_1 \times \mathrm{I}_2}$$

Where $Q_{\mathfrak{b}}^{\tilde{y},Disc,i}$ are N i.i.d random variables with the same ditribution as $Q_{\mathfrak{b}}^{\tilde{y},Disc}$.

Bibliography

[1] Frédéric Abergel, Côme Huré, and Huyên Pham. Algorithmic trading in a microstructural limit order book model. *arXiv preprint arXiv:1705.01446*, 2017.

[2] Frederic Abergel and Aymen Jedidi. A Mathematical Approach to Order Book Modeling. *Social Science Research Network Working Paper Series*, January 2011.

[3] Frederic Abergel and Aymen Jedidi. Long time behaviour of hawkes process-based limit order book. *Social Science Research Network Working Paper Series*, 2015.

[4] Frédéric Abergel, Charles-Albert Lehalle, and Mathieu Rosenbaum. Understanding the stakes of high-frequency trading. *The Journal of Trading*, 9(4):49–73, 2014.

[5] Aurélien Alfonsi and Pierre Blanc. Extension and calibration of a Hawkes-based optimal execution model, June 2015.

[6] Robert Almgren. Optimal Trading in a Dynamic Market. Technical Report 2, 2009.

[7] Robert Almgren and Neil Chriss. Value under liquidation. *Risk*, 12(12):61–63, 1999.

[8] Marco Avellaneda and Sasha Stoikov. High-frequency trading in a limit order book. *Quantitative Finance*, 8(3):217–224, 2008.

[9] N Baradel, B Bouchard, and Ngoc Minh Dang. Optimal control under uncertainty and bayesian parameters adjustments. *arXiv preprint arXiv:1604.06340*, 2016.

[10] Guy Barles and Panagiotis E Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic analysis*, 4(3):271–283, 1991.

[11] Bayer, C., Horst, U., and Qiu, J. . A functional limit theorem for limit order books with state dependent price dynamics. 2015.

[12] Paul Besson, Stéphanie Pelin, and Matthieu Lasnier. To cross or not to cross the spread: that is the question! Technical report, Kepler Cheuvreux, Paris, June 2016.

[13] Bruno Bouchard, Ngoc-Minh Dang, and Charles-Albert Lehalle. Optimal control of trading algorithms: a general impulse control approach. *SIAM J. Financial Mathematics*, 2(1):404–438, 2011.

[14] Álvaro Cartea, Sebastian Jaimungal, and José Penalva. *Algorithmic and High-Frequency Trading (Mathematics, Finance and Risk)*. Cambridge University Press, 1 edition, October 2015.

[15] Lehalle Charles-Albert and Mounjid Othmane. Limit Order Strategic Placement with Adverse Selection Risk and the Role of Latency. March 2016.

[16] Rama Cont and Adrien De Larrard. Price Dynamics in a Markovian Limit Order Book Market. *SIAM Journal for Financial Mathematics*, 4(1):1–25, January 2013.

[17] Rama Cont and Adrien De Larrard. Price dynamics in a markovian limit order market. *SIAM Journal on Financial Mathematics*, 4(1):1–25, 2013.

[18] Rama Cont, Sasha Stoikov, and Rishi Talreja. A Stochastic Model for Order Book Dynamics. *Social Science Research Network Working Paper Series*, September 2008.

[19] David Evangelista, Olivier Guéant, and Douglas Vieira. New closed-form approximations in multi-asset market making. *Preprint*, 2017.

[20] L. Fahrmeir and H. Kaufmann. Consistency and asymptotic normality of the maximum likelihood estimator in generalized linear models. *The Annals of Statistics*, 13(1):342–368, 1985.

[21] J. Doyne Farmer, Austin Gerig, Fabrizio Lillo, and Henri Waelbroeck. How Efficiency Shapes Market Impact. *Social Science Research Network Working Paper Series*, March 2013.

[22] Thierry Foucault. Order flow composition and trading costs in a dynamic limit order market1. *Journal of Financial Markets*, 2(2):99–134, 1999.

[23] Sanford J. Grossman and Merton H. Miller. Liquidity and market structure. *The Journal of Finance*, 43(3):617–633, 1988.

[24] Olivier Guéant. *The Financial Mathematics of Market Liquidity: From optimal execution to market making*, volume 33. CRC Press, 2016.

[25] Olivier Guéant. Optimal market making. *Applied Mathematical Finance*, 24(2):112–154, 2017.

[26] Olivier Guéant, Charles-Albert Lehalle, and Joaquin Fernandez-Tapia. Dealing with the inventory risk: a solution to the market making problem. *Mathematics and Financial Economics*, 4(7):477–507, September 2013.

[27] Fabien Guilbaud and Huyen Pham. Optimal high-frequency trading with limit and market orders. *Quantitative Finance*, 13(1):79–94, 2013.

[28] Thomas S. Y. Ho and Hans R. Stoll. The dynamics of dealer markets under competition. *The Journal of Finance*, 38(4):1053–1074, 1983.

[29] Weibing Huang, Charles-Albert Lehalle, and Mathieu Rosenbaum. Simulating and analyzing order book data: The queue-reactive model. *Journal of the American Statistical Association*, 10(509), December 2015.

[30] Weibing Huang and Mathieu Rosenm. Ergodicity and diffusivity of Markovian order book models: a general framework, May 2015.

[31] Aymen Jedidi. *Stochastic order book modelling*. PhD thesis, Ecole Centrale Paris, 2014.

[32] Albert P. Kyle. Continuous Auctions and Insider Trading. *Econometrica*, 53(6):1315–1335, 1985.

[33] Aimé Lachapelle, Jean-Michel Lasry, Charles-Albert Lehalle, and Pierre-Louis Lions. Efficiency of the Price Formation Process in Presence of High Frequency Participants: a Mean Field Game analysis. *Mathematics and Financial Economics*, 10(3):223–262, June 2016.

[34] Peter Lakner, Josh Reed, and Sasha Stoikov. High Frequency Asymptotics for the Limit Order Book. Technical report, New York University Stern School of Business, July 2013.

[35] Alexander Lipton, Umberto Pesavento, and Michael G. Sotiropoulos. Trade arrival dynamics and quote imbalance in a limit order book, December 2013.

[36] Anna Obizhaeva and Jiang Wang. Optimal Trading Strategy and Supply/Demand Dynamics. *Social Science Research Network Working Paper Series*, February 2005.

[37] Christine A. Parlour. Price Dynamics in Limit Order Markets. *The Review of Financial Studies*, 11(4):789–816, 1998.

[38] Ioanid Ro¸su. A dynamic model of the limit order book. *Review of Financial studies*, 22:4601–4641, 2009.

[39] E. Smith, Doyne J. Farmer, L. Gillemot, and S. Krishnamurthy. Statistical Theory of the Continuous Double Auction. *Quantitative Finance*, 3(6):481–514, 2003.

[40] Ioane M. Toke. Stationary distribution of the volume at the best quote in a Poisson order book model, February 2015.

Résumé

Cette thèse propose des modèles et des méthodes pour étudier le contrôle du risque dans de larges systèmes financiers. Nous proposons dans une première partie une approche structurelle : nous considérons un système financier représenté comme un réseau d'institutions connectées entre elles par des interactions stratégiques sources de financement mais également par des interactions qui les exposent è un risque de contagion de défaut. La nouveauté de notre approche réside dans le fait que ces deux types d'interaction interfr̀ent. Nous proposons des nouvelles notions d'équilibre pour ces systèmes et étudions la connectivité optimale du réseau et le risque systémique associé. Dans une deuxième partie, nous introduisons des mesures de risque systémique définies par des équations différentielles stochastiques rétrogrades dirigées par des opérateurs à champ moyen et étudions des problèmes d'arrêt optimal associés. La dernière partie aborde des questions de liquidation optimale de portefeuilles.

Abstract

This thesis presents models and methodologies to understand the control of systemic risk in large systems. We propose two approaches. The first one is structural : a financial system is represented as a network of institutions. They have strategic interactions as well as direct interactions through linkages in a contagion process. The novelty of our approach is that these two types of interactions are intertwined themselves and we propose new notions of equilibria for such games and analyze the systemic risk emerging in equilibrium. The second approach is a reduced form. We model the dynamics of regulatory capital using a mean field operator : required capital depends on the standalone risk but also on the evolution of the capital of all other banks in the system. In this model, required capital is a dynamic risk measure and is represented as a the solution of a mean-field BDSE with jumps. We show a novel dual representation theorem. In the context of mean-field BSDEs the representation gives yield to a stochastic discount factor and a worst-case probability measure that encompasses the overall interactions in the system. We also solve the optimal stopping problem of dynamic risk measure by connecting it to the solution of reflected mean-field BSDE with jumps. Finally, We provide a comprehensive model for the order book dynamics and optimal Market making strategy appeared in liquidity risk problems.

Mots Clés

Risque systémique, réseaux financiers, graphes aléatoires, contagion de défauts, contrôle optimal, EDSRs à champs moyen avec sauts.

Keywords

Systemic risk, financial networks, random graphs, default contagion, optimal control, BSDEs with jumps, Mean field BSDEs.